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OPTICAL DETERMINATION OF THE DENSITY OF He^3

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An interference method is described for determining the density of gaseous and liquid He^3. It is found that for He^3 the molar polarization $A = 0.123$ cm^3/mole. It is shown that at temperatures and pressures below the critical values the equation of state for gaseous He^3 is described by the formula $p/\rho = 27.35\,T - 2.3 \times 10^3\rho + 1.8 \times 10^4\rho^2$, where p/ρ is expressed in atm cm^3/g. The compressibility curves for liquid He^3 are determined. For the critical point the values $T_c = 3.38 \pm 0.03$°K, $p_c = 930 \pm 20$ mm Hg, and $\rho_c = 0.041 \pm 0.001$ g/cm^3 are obtained.

FOR the determination of the density of He^3, in both the liquid and the gaseous phase, one can employ the formula for the molar polarization

$$\frac{3}{4\pi}\,\frac{n^2-1}{n^2+2}\,\frac{M}{\rho}=A=\text{const.}$$

In accordance with the measurements of the Cuthbertsons,[1] the value of the molar polarization for gaseous He^4 under normal conditions is 0.124 cm^3/mole. From the measurements of Johns and Wilhelm[2] this constant is $A = 0.117$ cm^3/mole for liquid He^4 in the vicinity of 2°K at $\lambda = 5461$ A. It could be assumed that for He^3 as well the molar polarization would to within a few percent be constant and equal to the molar polarization of He^4. Since for helium n differs only very slightly from 1, we obtain, by setting $n = 1 + \Delta$ and expanding A in a series in Δ, the expression

$$A=\frac{M}{2\pi\rho}\,\Delta\left(1-\frac{\Delta}{6}+\cdots\right).$$

For liquid He^3, Δ is less than 0.02; therefore to within 0.4% the density $\rho = M\Delta/2\pi A$. Using this relation it is possible to determine the density of He^3 by measuring Δ from the number of interference fringes passing as an optical wedge is filled with He^3. Fig. 1 represents an apparatus constructed for this purpose.

A Dewar vessel 1 filled with liquid He^4 served as a cryostat, within which was placed a cylindrical glass tube 2. At its top the tube terminated in a flange, and was sealed off, both from the Dewar vessel and the atmosphere, with the aid of a rubber ring and a plane glass window. At its lower end the tube had an internal flange ground accurately perpendicular to the axis of the tube. The cell 3, a cross-section of which is illustrated on a larger scale at the bottom of Fig. 1, was attached to the ground portion with BF-4 cement. The cell was assembled, also with BF-4 cement, from a plane-parallel glass spacer 4, two plane-parallel glass flats 5, and a platinum foil 6. The spacer was 2.9 mm thick, with an aperture 4 mm wide and 17 mm long. The glass flats, 4 mm thick and 17 mm wide, were cemented to the top and bottom of the spacer. The upper flat was made 5 mm longer than the spacer and lower flat, in order that the interference at the very edge of the cell could be observed after it was cemented to the tube. The bottom face of the cell was frosted and blackened, so that reflection from the lower face would not affect the interference within. The opening at the end of the cell was closed with a bent piece of platinum foil 0.2 mm thick. This platinum foil insured thermal contact between the interior and the surrounding helium, at least in the immediate vicinity of the foil. Into the rear portion of the spacer was sealed a platinum tube 7, through which He^3 was admitted to the cell. A German silver tube 8, 1.4 × 2 mm in diameter, was soldered to the platinum tube. The pressure of the He^3 within the cell was measured with a mercury manometer 9. With the aid of a telescope 10 and a low-pressure mercury lamp 11, fringes of equal optical thickness were observed between the plane-parallel plates 5. Thirteen fringes were included within the length of the cell. The lamp produced its most intense illumination at a wave length $\lambda = 5461$ A. The clearest interference pattern was obtained for a light-source distance of 4 — 5 m, and remained adequately clear for a lens diaphragm opening of 1 cm.

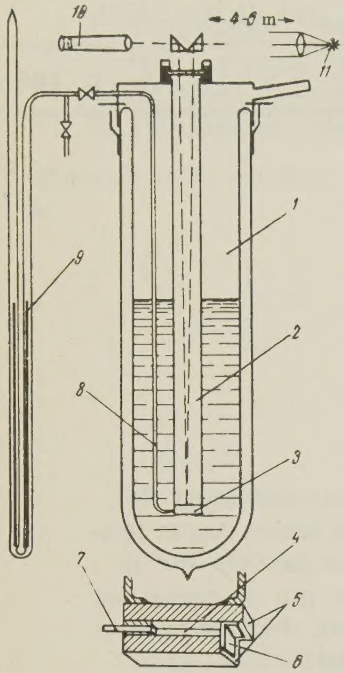

FIG. 1. Apparatus for determining the index of refraction of He^3.

A displacement of k interference fringes within the cell corresponded to a change in the index of refraction $\Delta = k\lambda/2d$, where d is the distance between the plane-parallel plates at the point at which the fringes are observed; the density of the helium can consequently be determined from the formula $\rho = (M\lambda/4\pi dA)k$. The distance between the flats near the foil was found after assembly to be $d = 2.93$ mm; the density of liquid He^3 in equilibrium with its vapor at $T = 2°K$, from the measurements of Kerr,[3] is $\rho = 0.078\ g/cm^3$, which corresponded to the passage of a number of fringes $k = 216$. From this we obtain $A = 0.123\ cm^3/mole$, and $\rho = 3.62 \times 10^{-4}k\ g/cm^3$. Using this formula, determinations of the density were carried out for both the liquid and the gaseous phase of He^3.

The experiments for the determination of the density of the gaseous phase were conducted at constant temperature. The pressure of the He^3 within the cell was progressively raised or lowered, and the reading of the manometer 9 was recorded each time a full fringe passed across a mark immediately adjacent to the platinum foil. As the condensation pressure was reached the motion of the fringes stopped, and liquid appeared within the cell. If observations on the compressibility were made in the vicinity of the critical point, then in addition to an increase in the rapidity of motion of the fringes marked fluctuations in the density were noted, the fringes behaving in an animated fashion.

The accuracy of the counting was limited primarily by the error in determining the moment at which the fringes passed, and corresponded to 0.2 — 0.3 of a fringe, or to $\Delta\rho \approx 10^{-4}\ g/cm^3$. Since for He^3 at extremely low densities deviations from the ideal gas law are hardly to be expected, the procession of the fringes at the foil was observed in the most convenient position for counting. p/ρ was then plotted graphically as a function of ρ, a fraction of the value of k being chosen such that extrapolation of the curve to zero density gave a value for p/ρ corresponding to that for an ideal gas.

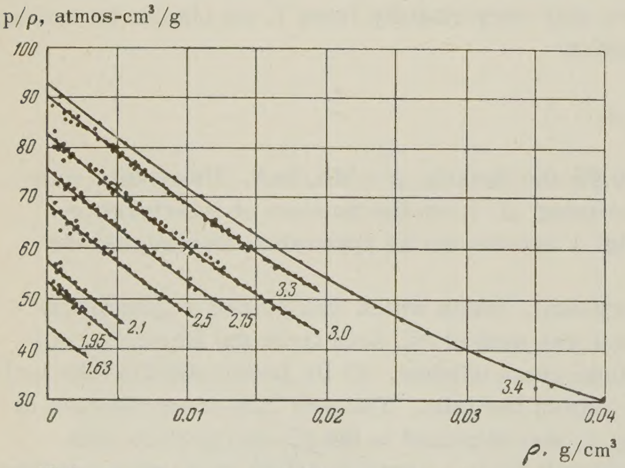

FIG. 2. Compressibility curves for gaseous He^3. The temperature is indicated on the curves in °K.

The compressibility curves for gaseous He^3 are presented in Fig. 2. The points indicate the experimental data, while the solid curves correspond to the equation

$$p/\rho = 27.35T - 2.3\cdot 10^3\rho + 1.8\cdot 10^4\rho^2,$$

in which p/ρ is expressed in atm-cm³/g. For comparison, the crosses in the graph represent some of the data of Keller,[4] who determined the density of He^3 by the customary method of measuring volumes, pressures and temperatures. Other data by Keller are not included, since his measurements terminated at densities below 0.01 g/cm^3 and were carried out at temperatures differing from those on the curves of Fig. 2, and when recomputed agree with the latter to within the accuracy of the measurements. As is evident from Fig. 2, the state curve for gaseous He^3 at temperatures and pressures below the critical values may to a sufficient degree of accuracy be described by an equation having three constant coefficients, the first of which corresponds to the ideal gas coefficient.

The density curve for gaseous helium in equilibrium with the liquid phase was determined both by fixing the moment at which the procession of fringes ceased during compression at various given temperatures and by observation of the variation of the equilibrium density with temperature. In the latter case, the displacement of the fringes was observed in the immediate vicinity of the foil as the temperature was lowered, under the condition that the liquid phase be visible within the cell at some distance from the foil. For this purpose, the apparatus was inclined laterally through 9°. The results of the measurements are presented in Fig. 3 (lower curve). The points correspond to the measurements described above, the crosses are taken from the data of Kerr.[3]

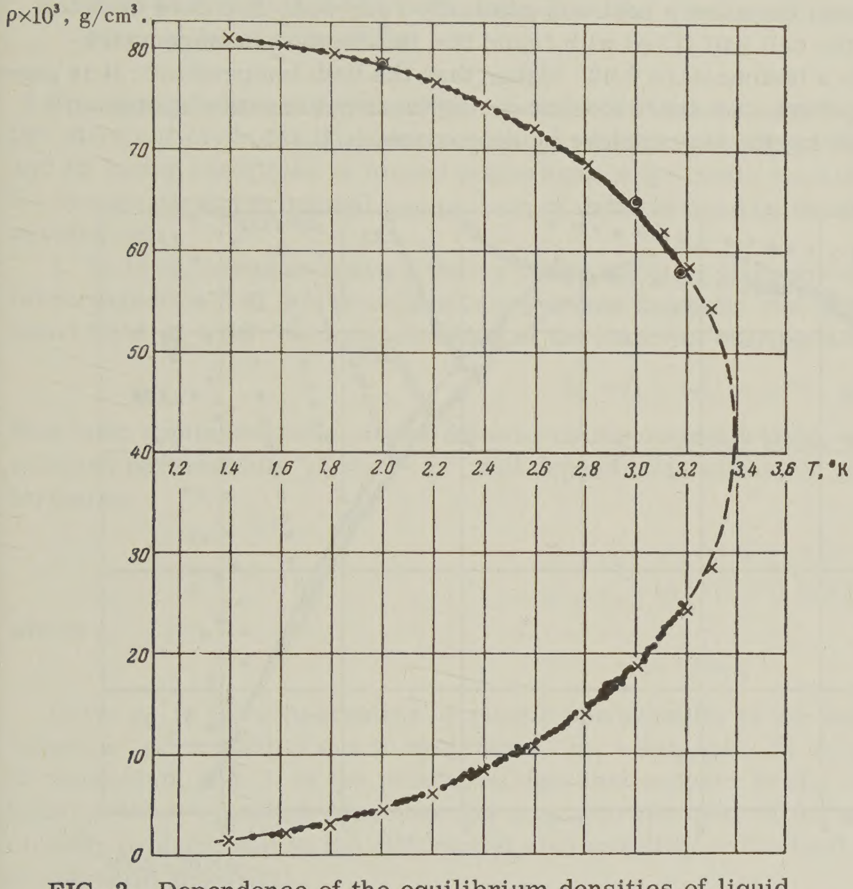

FIG. 3. Dependence of the equilibrium densities of liquid and gaseous He^3 upon temperature.

In order to determine the density of liquid He^3 in equilibrium with the gaseous phase, an uninterrupted count of the fringes was maintained during the following operations. First, the cell was filled at a temperature of 3.37°K to a pressure above the critical value (996 mm Hg); then the temperature was lowered to 2.00°K and at this temperature the pressure within the cell was reduced to the equilibrium value for the vapor, which was observed by the appearance of bubbles. In this case the apparatus was tilted in such a way that at the foil the cell was completely filled with liquid, while further away the boundary between the liquid and gaseous phases was visible. During the process described above the equilibrium density of liquid He^3 at 2.00°K, which, as indicated above, was taken to be 0.078 g/cm^3, corresponded to the passage of 216 fringes. All of the remaining liquid He^3 density measurements were referred to this point. A curve showing the dependence of the density of liquid He^3 in equilibrium with its vapor was obtained by slowly reducing the temperature (about 2 hr), under the condition that the boundary between the liquid and gaseous phases should always be visible within the cell. The results of these measurements are shown in Fig. 3 (points along the upper curve); the crosses are from the data of Kerr.[3] The large circles on the curve represent points measured during the course of the above-described continuous transition from the gaseous phase to the liquid through the critical point.

The most careful measurements correspond to $k = 216$, $\rho = 0.078$ g/cm^3, and $T = 2.00$°K; the remaining two points to $k = 180$, $\rho = 0.065$ g/cm^3, and $T = 3.01$°K, and to $k = 161$, $\rho = 0.058$ g/cm^3, and $T = 3.16$°K.

For the critical point the values $T_c = 3.38 \pm 0.03$°K, $p_c = 930$ mm Hg, and $\rho_c = 0.041 \pm 0.001$ g/cm^3 are obtained.

Using this same apparatus, measurements of the compressibility of liquid He^3 were carried out at various temperatures. In these experiments the temperature of the bath was held constant and the displacement of the fringes in the immediate vicinity of the foil was observed as the pressure within the cell was increased. The results of these measurements are presented in Fig. 4. From these data on the compressibility the velocity of sound in He^3 should vary from 170 m/sec at 1.6°K to 80 m/sec at 3°K.

The state diagram for He^3 in the coordinates ρ and p is given in Fig. 5. The points represent data from individual measurements. The solid lines in the diagram show the course of the variations in the density of liquid He^3 as the temperature was varied from 2.1° and from 3.22°K at a rate of approximately 1° per hour. During this time there was no pumping on the helium vapor, the tube extending upwards was not cooled by evaporating helium, the heat input increased, and a non-equilibrium situation was observed. With pumping and the establishment of a constant temperature, the excess heat was gradually removed. It should be noted that with the cell half filled with liquid the He^3 vapor pressure corresponded to a temperature 0.02° higher than the bath temperature; it is possible, therefore, that there existed during the experiments a systematic error lowering the temperature by approximately 0.01°.

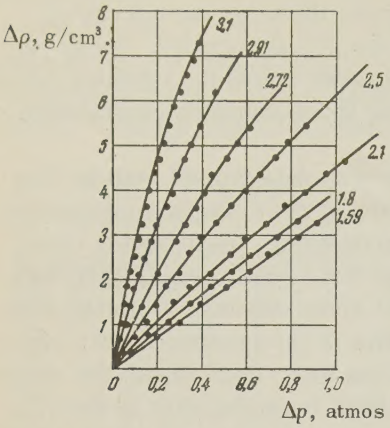

FIG. 4. Compressibility curves for liquid He^3. Numbers on the curves give T in °K.

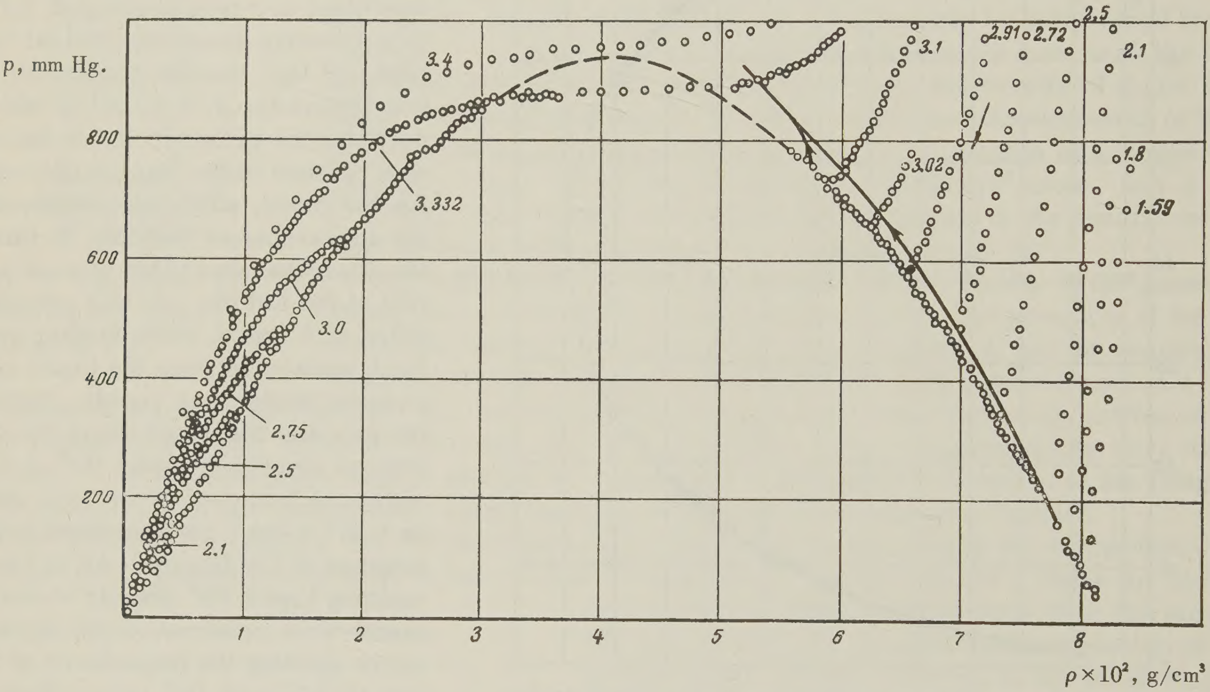

FIG. 5. State diagram for He^3 in coordinates ρ and p.

The optical method thus described may, after adaptation for measurements at higher pressures, be used successfully for measurements of the densities of other gases and liquids.

In conclusion, I take this opportunity to express my gratitude to N. I. Kondrat'ev, A. I. Filimonov, and I. A. Uryutov, who assisted me in the performance of these experiments.

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EFFECT OF MAGNETIC VISCOSITY ON THE FREQUENCY PROPERTIES OF FERRITES

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The magnetic viscosity of a series of nickel-zinc ferrites has been studied under conditions of free and of forced variation of magnetization. For the first time, the viscosity properties have been measured by both methods on the same specimens. An estimate is made of the viscous friction constant.

THE effect of magnetic viscosity in metallic and semiconducting ferromagnetics is ordinarily studied by two different methods: (1) under conditions of free variation of the magnetization (aperiodic conditions), and (2) under conditions of forced magnetization (periodic conditions, sinusoidal field with frequency f). No systematic experimental comparison of data obtained by these different methods has hitherto been carried out.

1. In 1928, Arkad'ev gave a theory of the effect of magnetic viscosity on the frequency properties of ferromagnetics.[1] If a ferromagnetic possesses magnetic viscosity, i.e., if in response to a suddenly altered field $H + \Delta H$ the magnetization of the material changes according to

$$\Delta I = I_1 + I_2(1 - e^{-t/\tau}), \tag{1}$$

then upon application of a sinusoidal alternating magnetic field, with amplitude H_0 and frequency f, the complex permeability $\mu' = \mu - i\rho'$ will depend on frequency in accordance with the following relaxational formulas:

$$\mu = \mu_0 + m_1 / [1 + (f / f_u)^2]; \tag{2}$$

$$\rho' = \rho'_h + m_1 f / [1 + (f / f_u)^2], \tag{3}$$

where

$$f_u = 1 / 2\pi\tau. \tag{4}$$

Here μ_0 is the conservative or elastic permeability of the material at very high frequency, ρ'_h is the "viscous" permeability due to hysteresis, $m_1 = 4\pi I_2 / \Delta H$, f_u is a critical frequency at which ρ' reaches its maximum, and τ is the relaxation time that appears in (1). As has been shown by one of us,[2] the relaxation time τ, which describes the magnetic viscosity of the material, is at constant temperature directly proportional to the differential susceptibility in the final state of the ferromagnetic (the first law of magnetic viscosity),

$$\tau = A\chi_d(H), \tag{5}$$

where A is a constant, independent of the field H. Consequently the critical frequency according to (4) will be

$$f_u = 1/2\pi A\chi_d(H). \tag{6}$$

Becker[3] calculated the damping effect of microscopic eddy currents for metallic ferromagnetics. In weak fields, where the dominant process is reversible displacement of the walls between domains, the frequency dependence of the initial permeability, according to Becker, is of relaxational character, since this author neglects the effect of wall "inertia". The relaxation frequency in this case is inversely proportional to the friction constant Ω, which is determined chiefly by the electrical conductivity of the ferromagnetic. The relaxation frequency is furthermore inversely proportional to the initial susceptibility. Consequently the relaxation time for magnetization of a metallic ferromagnetic in a weak field is directly proportional to the product $\Omega\chi_a$. From Becker's work it follows that

$$\tau = (\Omega l / 3I_s)\chi_a. \tag{7}$$

No. of specimen	Chemical composition mol %		t° Sintering temperature °C	Static data					Aperiodic conditions		Frequency properties	
	$NiOFe_2O_3$	$ZnOFe_2O_3$		μ_a, G/Oe	I_s, G	H_c, mOe	ρ, $\Omega\cdot$cm	χ_d max, G/Oe	τ_{max}, μsec	τ/χ_{jump}, sec Oe/G	ΩS, Oe sec	h_0, mOe
1	15.3	34.7	1350	1700	160	70	$3\cdot10^5$	1150	20	$2.7\cdot10^{-8}$	$3.2\cdot10^{-6}$	35
2	17.4	32.6	1275	1250	125	130	$4\cdot10^5$	340	7	$3.3\cdot10^{-8}$	$0.9\cdot10^{-6}$	85
3	17.4	32.6	1320	1900	160	85	$0.2\cdot10^5$	880	25	$4\cdot10^{-8}$	$1.9\cdot10^{-6}$	60
4	17.4	32.6	1375	2250	160	50	$0.9\cdot10^4$	1140	40	$4.3\cdot10^{-8}$	$2.2\cdot10^{-6}$	45
5	17.4	32.6	1400	2920	180	60	$0.2\cdot10^5$	830	21	$4.3\cdot10^{-8}$	$1.4\cdot10^{-6}$	40

Becker[3] also calculated a critical frequency for irreversible jumps, likewise brought about by the presence of magnetic friction. Let the mean jump distance of a wall, for quasi-static magnetization, be S ($S \leq \ell \sim 10^{-4}$ cm); and let the mean value of the critical field, below which jumps are impossible, be h_0. Then the jump speed v in average fields conforms to the known law

$$\Omega v = H - h_0. \tag{8}$$

The distance S is traversed by a wall in time $\tau' = S\Omega/(H - h_0)$. Therefore Becker estimates the critical frequency for such irreversible jumps as

$$\omega_{cr} = (H - h_0)/\Omega S. \tag{9}$$

For frequencies exceeding ω_{cr}, a wall executing a jump does not have time to traverse the full distance S during a half period; therefore the permeability decreases with increasing frequency.

Numerous experiments on the study of magnetic viscosity under conditions of free magnetization, conducted in our laboratory, have shown that large magnetic viscosity is observed precisely in the range of irreversible jumps, that is in average fields. In weak and very strong fields, the magnetic viscosity is appreciably smaller. The nature of the time dependence of the magnetization in average fields has been insufficiently investigated theoretically. At present there are only the formal activation theory of Street and Woolley (1948) and the fluctuation theory of Néel (1951) for the time dependence of magnetization in this range of fields.

If viscosity is especially large in the range where hysteresis most clearly exhibits itself, then probably both phenomena (hysteresis and magnetic viscosity) are of the same nature. Kondorskii[4] established that the shape of the hysteresis curve of a polycrystalline ferromagnetic is determined by the magnetic interaction of the crystallites or domains. Such magnetic interaction explains the mode of dependence of the differential irreversible-jump susceptibility upon the external field. Apparently a calculation of the magnetic interaction of the particles can give a definite correlation between the friction constant and the magnetic viscosity, which is proportional to the differential jump susceptibility.

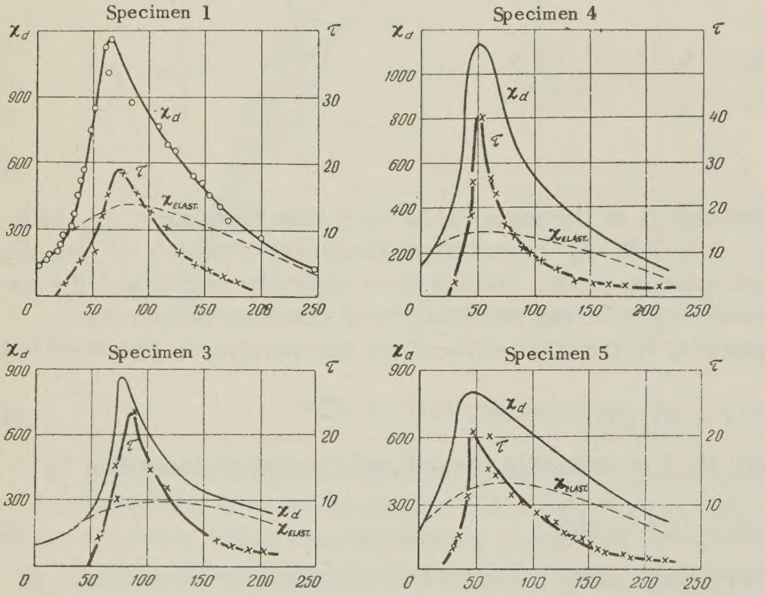

FIG. 1. Dependence of relaxation time τ (μ sec) and of differential susceptibility χ_d on magnetic field (mOe). The dashed curve is the elastic component of the susceptibility, χ_{elast}.

2. We have performed experiments with the object of observing the effect of the magnetic viscosity of ferrites on their frequency properties. Five toroidal specimens of nickel-zinc ferrites were studied; their compositions, sintering temperatures (the sintering lasted four hours), and magnetic properties are given in the table. The measurement of magnetic viscosity was carried out by a method that was developed by Telesnin and Lednev[5] and that uses a pulsed oscillograph. The measurements were made by symmetric pulsed magnetization reversal of the specimen from $+H_0$ to $-H_0$. Such conditions are convenient for comparison with frequency data. Just as in Ref. 5,

large magnetic viscosity was observed only in the range of irreversible jumps. The emf vs time curve could be approximated with an exponential.

Curves that show the dependence of the relaxation time τ, characteristic of the magnetic viscosity, on the magnetic field H_0 are given in Fig. 1. The results of the experiments at room temperature show that (5) is well satisfied on all the specimens, provided χ_d is interpreted as the irreversible component χ_{jump} of the susceptibility, corresponding to magnetization not by elastic displacement, but by sudden jumps ("jump" in the subscript) of walls from one position to another.

The total differential susceptibility, measured by the usual ballistic method, consists of two components — the elastic susceptibility χ_{elast} (shown in dashes in Fig. 1), and the jump susceptibility χ_{jump} — so that $\chi_d = \chi_{elast} + \chi_{jump}$. Such an analysis of the susceptibility corresponds to Arkad'ev's expression (1), in which the change of magnetization is separated into a "rapid" and a "viscous" part. In weak fields (less than the critical field h_0), where there are no jumps, the magnetic viscosity is negligible (less than 1 μsec), though the initial susceptibility is very large ($\chi_a \sim 200$).

The values of the constant $\tau/\chi_{jump} = A$ for all the specimens, at room temperature, are given in the table. Also given are the maximum values of χ_d and τ.

As can be seen from Fig. 1, the critical frequencies (4), $f_u \sim 1/\tau$, should decrease rapidly upon change of the field H_0 and should lie in the interval 1.5 to 15 kcs, i.e. in the audio range. Therefore an investigation with sinusoidal fields was carried out in the range 400 cps to 60 kcs.

The permeabilities μ and ρ' were determined by means of a differential transformer; one of its arms contained the specimen, and the other contained a standard resistance R and inductance L. To decrease the effect of harmonics, additional resistance was inserted in the arms, and the measurement circuit of the bridge contained a filter and a selective amplifier. Since skin effect is absent in ferrites, the permeabilities were determined by the formulas

$$\mu = L/L_0, \quad \rho' = \Delta R/2\pi f L_0, \tag{10}$$

where L_0 is the inductance of the empty winding of the specimen, and R is the effective resistance due to losses in the core. Figs. 2 and 3 show the dependence of μ and ρ' on frequency for specimen 3, at constant amplitude of the magnetic field H_0. Similar dependence was found for all five specimens. In all cases with $H_0 > h_0$ (jumps occurring), the frequency dependence of ρ' shows a maximum at some frequency f_{cr}.

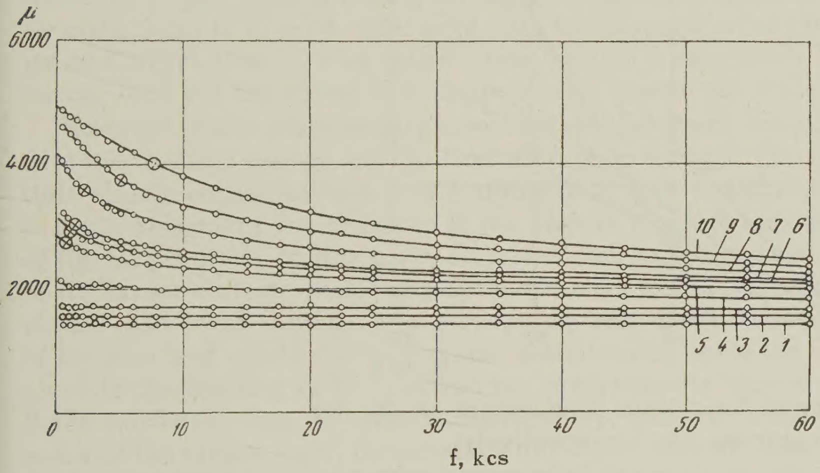

FIG. 2. Dependence of the real part of the magnetic permeability of specimen no. 3 on frequency at constant amplitude H_0 of the magnetic field. The values of H_0 (in millioersteds) for the various curves are: 1, 9; 2, 22; 3, 36; 4, 56; 5, 66; 6, 72; 7, 78; 8, 94; 9, 120; 10, 150.

In Fig. 2 it is clearly evident that the permeability $\mu \sim B/H$ consists of two components: an elastic component that varies only slightly with frequency in this range, and a jump permeability with an evident relaxational dependence on frequency. The circled crosses mark the permeabilities at the critical frequency. The dependence of the critical frequencies f_{cr} on the amplitude of the magnetic field (cf. Fig. 4) was quite different from (6). Instead of $f_u \sim 1/\tau$ we have a linear increase of f_{cr} with field H_0. This kind of dependence can be explained within the framework of Becker's theory. From the graphs it is easy to find the values of h_0, which determine the critical field and the quantity ΩS determined by (9). The corresponding values are given in the table.

The frequency dependence of the permeability of our specimens shows that on increase of frequency in the audio range, the irreversible jumps responsible for magnetic viscosity are suppressed because of magnetic friction, in accordance with (9), even before attainment of the frequencies (4). The calculated condition for observation of relaxation at frequency f_u follows from a comparison of (9) and (4):

$$(H_0 - h_0)/\Omega S > 1/\tau \quad \text{or} \quad (H - h_0)\chi_{jump}(H_0) > \Omega S/A. \tag{11}$$

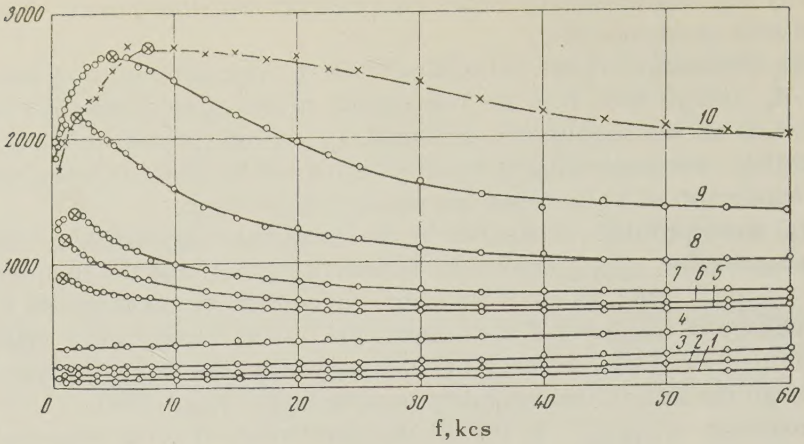

FIG. 3. Dependence of the imaginary part of the magnetic permeability of specimen no. 3 on the frequency at constant amplitude of the magnetic field H_0. The notation is the same as in Fig. 2.

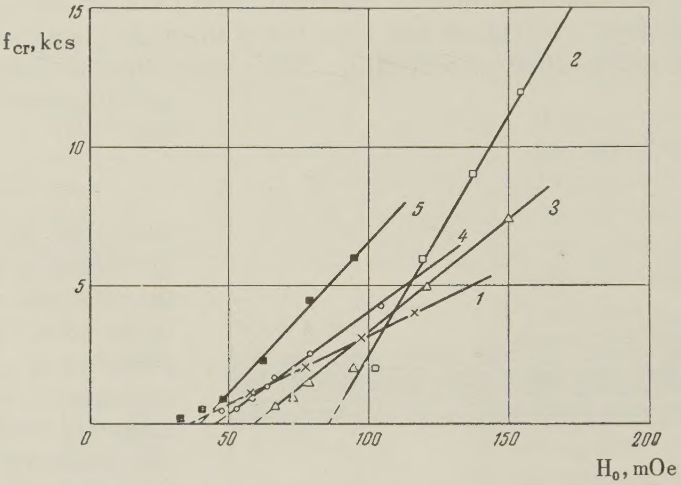

FIG. 4. Dependence of the critical frequencies on H_0 for specimens nos. 1 to 5 (curves 1 to 5 respectively).

In the case of our specimens, condition (11) is not fulfilled. Unfortunately, it is not possible to determine Ω or S separately from our experiments. However, if we take $S \sim 10^{-4}$ cm, then our data permit estimation of the friction constant Ω; it was very high, $\sim 3 \times 10^{-2}$ oersted-sec/cm. This is apparently to be attributed to the approximate character of the relation (9). Further study of the temperature dependence of magnetic viscosity and of the frequency properties in average fields can contribute to the elucidation of the nature of magnetic friction in ferrites.

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ION SCATTERING COEFFICIENT AS A FUNCTION OF COLLIDING-PARTICLE MASS RATIO

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The double-modulation method has been used to study secondary ion emission in the case in which the mass of the bombarding ion is greater than the mass of the target atom. It is shown that when Ni is bombarded by Cs ions or when Mo is bombarded by Ba ions no ions are observed at the "reverse" angle. When $V_i > \varphi$ secondary ion emission is not observed at any target temperature; when $V_i < \varphi$ the secondary ion emission which is observed at high target temperatures consists of ions which diffuse from the interior of the target and are evaporated at the surface.

IT has been shown by us[1-3] that there is a group of secondary ions when the mass of the bombarding ions m_2 is smaller than the mass of the target atoms m_1, which are characterized by high energies, the limiting value of which is determined by elastic scattering relations. It has also been shown that the dependence of the limiting energy on angle of incidence of the primary ion and emission angle of the secondary ion is in good agreement with the assumption of an elastic interaction between bombarding ion and target atom. These experiments furnish a reasonable basis for assuming that at least the limiting-energy ions are the result of a single elastic scattering of the primary ions on individual target atoms.

However, these experiments also indicate that there are secondary ions which have energies lying between thermal energy and the limiting higher energy. The interpretation of these ions in terms of multiple elastic collisions with target atoms is in need of additional verification. Hence it is of interest to study secondary ion emission in the case in which the mass of the bombarding ion is larger than that of the target atom. In this case, as is well known, for normal incidence of the primary ions on the plane of the target a single scattering of the primary ions at the reverse angle cannot occur. In practice there should also be no secondary ions resulting from multiple collisions since there is only a small probability of the required number of successive collisions in which the angle of deflection of the primary ion is close to the limiting value required for reversing the sign of the velocity; moreover, the velocity of these secondary ions is. small. Consequently, when the mass of the bombarding ion is larger than the mass of the target atom, the primary ion should, for the most part, penetrate the target to a substantial distance. Under these conditions there should be no secondary ion emission if the target is cold.

At high temperatures the targets are penetrated and the primary ions which are absorbed in the target enter into the thermal motion of the target atoms, thereby diffusing to the surface and subsequently being evaporated in the form of ions or neutral atoms. The ratio of the number of evaporated neutral particles to the number of positively charged particles is found to be in agreement with the well-known Saha-Langmuir relation:

$$n_+ / n_0 = A \exp\left(-\frac{V_i - \varphi}{kT}\varepsilon\right).$$

Thus, when $m_1 < m_2$, especially when $V_i < \varphi$ (V_i is the ionization potential of the ion, φ is the work function of the target), one expects that the secondary ions will comprise two ion groups — ions which are evaporated directly after adsorption and ions which, having penetrated the target to some depth, are evaporated after diffusion to the surface. Both of these ion groups will appear only at high target temperatures and will obey the usual relation for surface ionization.

Having these possibilities in view, we have considered two cases known to be different with respect to target-surface work function and ionization potential of the bombarding ion. We have studied scattering of positive Cs ions on Ni ($V_i < \varphi$) and Ba ions on Mo ($V_i > \varphi$).

The work was carried out using the experimental arrangement described in Ref. 2, under the same vacuum conditions. The measurements were performed by the double-modulation method used by us in studying the secondary processes at the surfaces of pure metals in the case $m_1 > m_2$.[2,4]

CESIUM ON NICKEL

In Fig. 1 is shown an oscilloscope photograph of the retardation curve (volt-ampere characteristic) for a pure nickel target bombarded by cesium ions with energies of 400 ev. On the same figure are shown retardation curves for an incandescent target (1350° K, Curve 1) and a cold target (Curve 2). To obtain quantitative measurements four pulses of the total ion current I_Σ (Curve 3) were taken on the same photograph.

It is apparent from Curve 1 that ion secondary-emission coefficient for the incandescent target is better than 90 percent. The energies of these ions are small: the secondary ions are completely inhibited by a rather small retarding field. Examination of Curve 2 of the same figure indicates that in the cold-target case the ions secondary-emission coefficient is extremely small, being about 6 — 7 percent (these ions apparently originate in surface impurities). Consequently, the overwhelming majority of secondary ions (when $m_1 < m_2$ and $V_i < \varphi$) appear as a result of evaporation of ions of the primary beam which have been adsorbed at the surface of the target.

Further verification of this description can be found in Fig. 2, which shows two retardation curves taken with a lower modulation frequency of the primary beam and a target temperature of 1300° K (Curve 1) and a temperature of 1100° K (Curve 2). At 1300° K the "lifetime" of the ions adsorbed at the target surface is so small that the desorption process is completed almost immediately after the primary ions are cut off. Hence the variation of the secondary current follows the pulses of the primary ion beam. A different pattern is observed at the lower target temperature: in this case the "lifetime" of the adsorbed ions is considerable and the desorption process requires a certain period of time after ions from the primary beam no longer enter the target. Hence, Curve 2 of this photograph does not reproduce exactly the π-shaped pattern of the applied pulses, but rises and falls in a way which suggests an exponential relation.

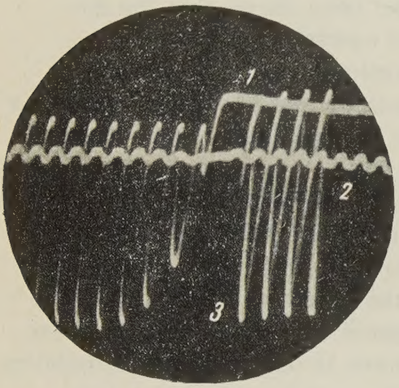

FIG. 1

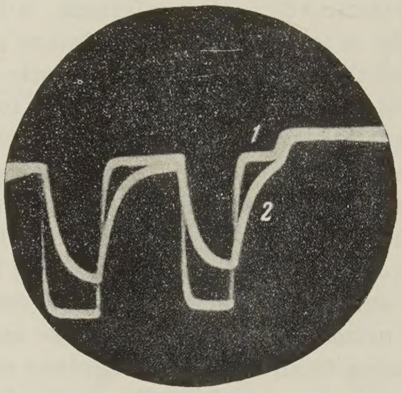

FIG. 2

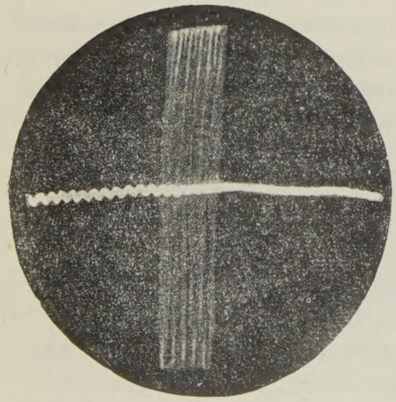

FIG. 3

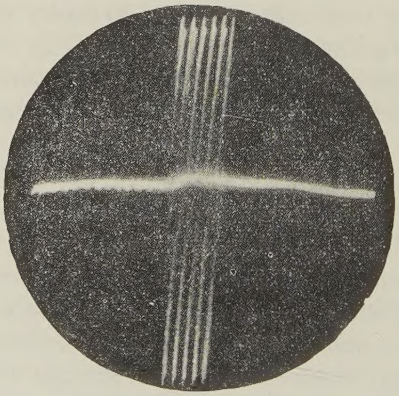

FIG. 4

The oscilloscope photographs in Fig. 1 and Fig. 2 also indicate the presence of secondary ions characterized by considerable delay, indicating fairly deep penetration of the primary ions into the target. The ions which penetrate the target to greater depths participate in the thermal motion, diffuse to the surface, and are then evaporated. Because of this effect a certain component of ion current from the target does not follow the modulation of the primary ion current. This phenomenon is clearly shown in the displacement of the base line of the oscillogram.

The experimental data which have been presented and calculations which have been performed indicate that the secondary ion emission from a pure nickel target at 1350° K bombarded by positive cesium ions consists mainly of evaporation (86 percent) and diffusion (14 percent) secondary ion currents. It should be emphasized that when a nickel target at high temperature is bombarded by cesium ions there is virtually no scattering or evaporation of cesium atoms.

Thus, it follows from the photographs shown in Fig. 1 and Fig. 2 that the secondary ion emission in the case $m_1 < m_2$ and $V_i < \varphi$, for example with Cs on Ni, does not take place at the expense of the energy of the primary ion beam. There is no change in the sign of the primary-ion velocity after interaction; consequently no ions are observed at the reverse angles. The secondary ions which are observed are ions which have penetrated into the target and then been evaporated from the surface because of the thermal motion due to the high temperature.

BARIUM ON MOLYBDENUM

It is also of interest to investigate the interaction between ions and target atoms in the case in which $m_1 < m_2$ but $V_i > \varphi$, i.e., when the surface ionization currents are vanishingly small.

In this work barium ions were used as bombarding particles. This element was chosen because its high ionization potential inhibits ion evaporation at elevated temperatures. Molybdenum was used as a target because it has a low work function as compared with nickel. In this case the difference between the work function and the ionization potential is known beforehand to be negative and the evaporation should consist mainly of neutral atoms.

In Fig. 3 is shown a photograph of the volt-ampere characteristic for a pure molybdenum surface at a temperature of 300° K bombarded by positive Ba ions with energies of 560 ev. It is apparent from the oscilloscope photograph that with a pure cold surface the secondary ion emission is negligibly small. A comparison of the secondary ion emission with the pulses of total primary ion current I_Σ (shown in the same photograph) gives a secondary ion emission coefficient $k < 1\%$.

A similar situation obtains at high target temperatures. In Fig. 4 is shown an oscilloscope photograph of the volt-ampere characteristic for the secondary currents when a molybdenum target at a temperature of 1300° K is bombarded by Ba ions with energies of 560 ev. In contrast to the case of Cs on Ni, in this case, as is apparent from the figure, there are no evaporated Ba ions at high target temperatures. Thus when a molybdenum surface is bombarded by Ba ions ($V_i > \varphi$, $m_1 < m_2$) the secondary ion emission is relatively small at any temperature of the bombarded target.

MECHANISM FOR SECONDARY ION EMISSION

On the basis of the foregoing sections and Refs. 1 — 4 we propose that when ions collide with a target the incident particles do not interact with the target as a whole, but rather with individual atoms of the crystal lattice. Under these conditions there is a strong analogy to the scattering of heavy particles in gases.

Thus a metal target can be considered as a system of gas atoms compressed into a very small volume. When $m_1 > m_2$, a large number of the observed scattered particles result from single collisions on surface atoms of the target. These particles have maximum energies, determined by the elastic-scattering relation

$$E = E_0 \frac{(m_1 - m_2)^2}{m_2^2 \left[\cos\beta + \sqrt{(m_1/m_2)^2 - \sin^2\beta}\right]^2}, \tag{1}$$

where E is the maximum energy of the secondary ions, E_0 is the energy of the primary ions, β is the angle between the incident direction and scattering direction of the ion, and m_1 and m_2 are the masses of the target atoms and the ions.

We have investigated the validity of this relation for a number of angles, in each case defining E as the limiting energy of the scattered particles at a given angle.[3] When the target-atom mass is smaller than the mass of the bombarding ion, the angle defining the maximum possible deflection of the bombarding particle is given by the expression:

$$\beta = \sin^{-1} (m_1/m_2). \tag{2}$$

For example, in the experiments in which the nickel target was bombarded by cesium atoms, after the first collision, having been driven into the target, the cesium ion cannot be deflected from its original direction by an angle greater than 26°. In the present experiments an extremely important conclusion can be derived from these relations: when $m_1 < m_2$ there will be no scattered primary ions. The experimental verification of this conclusion is a strong argument for the individual-interaction hypothesis proposed above.

On the other hand, when light ions bombard heavy targets, scattering at any angle can occur. This fact is also understandable from the point of view of single collisions. It follows from Eq. (2) that when $m_1 > m_2$ there is no limiting angle and ion scattering in any direction is possible. Complete experimental verification has also been obtained for this case.

A further indication that the primary ions penetrate into the target is given by the absence of "scattered" ions in the case in which nickel is bombarded by cesium. Both for the case $m_1 < m_2$ as well as $m_1 > m_2$ (Ref. 4) there is always penetration of the bombarding particle deep into the target. However, there is a difference in these two cases. When $m_1 < m_2$ all the bombarding particles penetrate to a certain depth and apparently have a definite range. In the second case there is strong scattering and only a small fraction of the primary particles reach the inside of the target. Whence it follows directly that the target atoms which are bombarded by ions with energies of several hundreds of volts are capable of being highly deformed, as a consequence of which there occur mainly individual collisions. These considerations are in agreement with existing interpretations. For example, using simple calculations Seitz[5] has shown that two protons with a relative energy of approximately 600 ev should approach to a distance of 2.4×10^{-11} cm, i.e., a distance equal to 1/100 of an atomic radius. Strong atomic deformation also occurs at lower energies: at an energy of approximately 20 ev, two atoms can collide so that the nucleus of one is found at the outermost radius of the other. At energies of the order of a hundred electron volts the atoms can approach to a distance of the order of 1/10 of a radius. Hence, in oblique collisions the incoming ion may penetrate into the metal just as a fast atom or ion penetrates a gas.

From this point of view the relatively slow secondary ions which reproduce the shape of the primary currents without "inertia" effects in the present experiments may be explained as follows. These slow inertialess ions are multiply-scattered primary particles which have experienced several collisions with deep-lying atoms of the target. The presence of slow ions with energies of approximately 0 — 10 ev can be explained only by collisions with atoms at a reasonable depth since the reduction of the particle energy by a factor of 10 or 100 requires at least 5 — 10 collisions. If these collisions were to take place at the surface, the particle, having an energy many times greater than the heat of evaporation, would have an extremely high probability for escaping from the target. Since the scattered ions with energies corresponding to a single elastic collision never total more than several percent, it may be assumed that the main fraction of the bombarding ions which experience oblique collisions penetrate into the target. These ions continue to move about in the target until they are no longer hindered by oblique collisions with target atoms or until they experience a head-on collision resulting in a loss of velocity which prevents escape.

In addition to those secondary ions which emerge without "inertia" effects, there are ions with very small (thermal) energies, the behavior of which is marked by extreme inertia effects. These ions are characterized by a definite delay with respect to the original ions and are observed only at rather high target temperatures. These ions are usually referred to as secondary ions. However, the presence of these ions is not directly related to the bombardment of the target by ions. This group of particles can be seen clearly only when $V_i < \varphi$ and appears as a result of thermal diffusion of the absorbed atoms and the subsequent surface ionization of these particles, which are converted into "adatoms."

The existence of an adsorbed layer of "adatoms," produced by virtue of the peculiar absorption of primary ions described above and diffusion, explains the time effects which have been observed both with galvanometers and oscilloscopes.

It should be noted that the appearance of a large number of "adatoms" associated with bombarding ions means that the target is covered by atoms with masses equal to the mass of the bombarding ions. In this case the parameters of the colliding particles are changed and instead of the condition $m_1 > m_2$ we have $m_1 = m_2$. As a result high-velocity ions are removed from the secondary-emission distribution. The ion velocity is completely lost in a head-on collision while in an oblique collision the ion enters into the target and may reappear only after several collisions, having lost a considerable part of its initial energy in the process.

CONCLUSIONS

On the basis of the experimental investigations reported here we propose the following:

1. From the example of positive Cs ions with Ni atoms for the case $V_i < \varphi$, $m_1 < m_2$ it has been shown that (a) there is virtually no secondary ion-electron emission from a pure cold (300° K) nickel surface or a heated (1350° K) nickel surface; (b) within the accuracy of these experiments the secondary ion emission contains no scattered Cs ions. At high temperatures the secondary ion emission from a pure nickel target consists only of evaporated ions which appear at the surface after diffusion.

2. From the example of positive Ba ions with Mo atoms for the case $V_i > \varphi$ and $m_1 < m_2$ it has been shown that (a) within an accuracy of about 1 percent there is no secondary ion-electron emission from a pure cold (300° K) molybdenum surface or a heated (1300° K) molybdenum surface; (b) within the accuracy of the experiments there is no secondary ion emission of any kind.

3. A qualitative examination of the results leads to the conclusion that secondary ion emission results from multiple individual scattering of ions which penetrate deep into the target and interact with target atoms.

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FORMATION OF NEGATIVE H⁻ *IONS IN COLLISIONS OF ELECTRONS WITH HYDROGEN MOLECULES*

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The formation of H^- ions has been observed when electrons collide with H_2 molecules. In the curve of H^- ion yield as a function of electron energy there is a sharp maximum at 14.5 ev, attributed to the resonance capture of electrons by the H_2 molecules. The formation of H^- ions at higher electron energies apparently results from the dissociation of H_2 molecules into positive and negative ions.

WHILE studying the ionization of hydrogen molecules by electron bombardment, Lozier[1] found negative ions present in his apparatus; these were formed in a narrow range of electron energy, with maxima occurring at 6.6 and 8.8 ev. Mass-spectrometric analyses carried out by Bleakney showed that they were atomic ions, H^-.

The number of ions so formed was extremely small, and Lozier suspected that they came, not from H_2 molecules, but from molecules of H_2O which might have entered the apparatus as an impurity in the hydrogen. And, in fact, when water vapor was added to the hydrogen these same negative ions were formed in much larger numbers. This established the fact that interactions between electrons and water vapor could give rise to H^- ions; but the possibility that these ions could also be formed by electrons colliding with H_2 atoms remained unexplored.

The aim of the present work was to study this second possibility under conditions where the masking effect of water vapor was removed as completely as possible, using a highly sensitive method of detecting the negative ions.

The experiments were conducted in glass apparatus (a mass spectrometer) with no stopcocks or massive metal parts. The ions were obtained by electron bombardment in an ion source of the usual type. The cathode was made of thoriated tungsten, and was kept below temperatures at which hydrogen could dissociate on its surface. The electron beam entered the ionization chamber through a 1 × 6 mm slit, and was kept from spreading by a magnetic field of about 150 gauss.

In order to make full use of the ions produced in the source, the ionization chamber slit and the entrance slit to the mass analyzer were both made 3 mm wide. The slit in front of the collector had a width of 3.5 mm. With this arrangement, the resolving power of the mass spectrometer was still sufficient to distinguish between ions of mass 16 and mass 18.

To measure ion currents at the exit of the mass spectrometer, an electron multiplier was used, with a multiplication factor of approximately 1000 for H^- ions with energies of 1000 ev. The output of the electron multiplier was connected to an electrometer amplifier with a sensitivity of 10^{-15} amp/mm. With this arrangement it was therefore possible to measure ion currents as low as 10^{-17} amp.

Special attention was paid to the purification of the hydrogen introduced into the ion source. Gaseous hydrogen was obtained from the liquid phase, filtered through a heated palladium tube, and then passed through a glass helix cooled with liquid nitrogen before entering the apparatus.

Mercury pumps were used to evacuate the apparatus, the mercury vapor being condensed in traps cooled with liquid nitrogen. Before the measurements, the entire apparatus was baked out at a temperature of 400°C for several hours.

A series of preliminary experiments was carried out in order to explain the presence of a "background" of H^- ions in the apparatus. After being pumped down to the limiting vacuum (but with no baking-out) the apparatus produced H^- ions when the electron accelerating potential in the ion source was between 6 and 13 volts. Upon baking out the apparatus, the number of H^- ions (arising, apparently, from the presence of traces of water in the apparatus) decreased to an unobservable value.

After removal of the residual H^- ions by prolonged baking of the apparatus, hydrogen was admitted into the ionization chamber. Its pressure could not be measured within the chamber itself; but a thermionic gauge sealed into the hydrogen inlet tube indicated a hydrogen pressure of a few times 10^{-4} mm Hg.

When hydrogen was admitted into the ionization chamber, H^- ions appeared in the apparatus in larger numbers and over a wider range of electron energies than in the case of the preliminary "background" experiments.

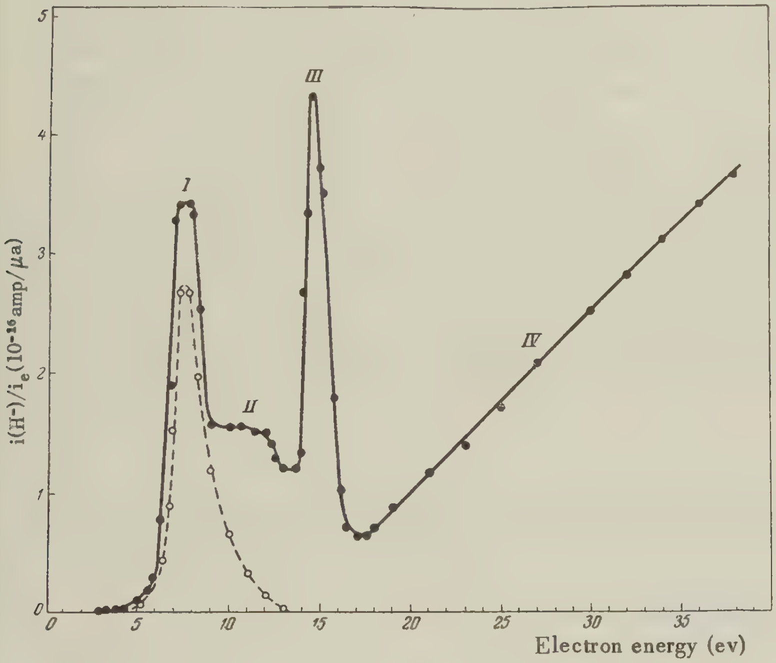
At a constant value of hydrogen pressure (on the order of 10^{-4} mm Hg) we measured the negative H^- ion currents at the mass spectrometer collector for electron accelerating voltages in the ion source of 2 to 40 volts. Simultaneously, for each value of accelerating voltage we measured the electron current in the ionization chamber. We calibrated the zero of the electron energy scale by measuring the yield curve for negative O^- ions from the collision of electrons with O_2 molecules. Using Thorburn's data[2] for the appearance potential of negative O^- ions from O_2, we determined the correction to be applied to the accelerating potential difference. This correction was found to be − 0.5 volts. We used the same value of the correction over the entire interval of electron energies studied.

The figure shows the curve of H^- ion yield obtained from our studies on hydrogen. Electron energy is plotted along the horizontal axis, in electron volts, including the zero-point correction. The ordinate is the ratio of H^- ion current $i(H^-)$, at the mass spectrometer collector, to the electron current i_e in the ion source.

In the region of electron energy investigated, the yield curve for H^- ions shows three maxima, marked I, II, and III on the figure, and a steadily rising portion on the right-hand side, labelled IV. In all these regions of the curve we have measured the dependence of the ion current $i(H^-)$ on the hydrogen pressure. At maximum III, and for a series of points in region IV, the H^- ion current varies linearly with the hydrogen pressure between 1×10^{-5} and 2×10^{-4} mm Hg. The height of maximum I was not proportional to the pressure. When the hydrogen was pumped out of the apparatus, maximum III and region IV completely disappeared at once. Maximum I (and, to a lesser extent, maximum II) could be observed for some time, even after the hydrogen had been removed from the apparatus.

Tests were made of the dependence of H^- ion current on the electron current density j_e in the ionization chamber. When j_e was between 0.15 and 3 $\mu a/mm^2$, this dependence was linear.

On the basis of the above observations, the curve in the accompanying figure can be explained in the following way. Maximum I is apparently not connected with the ionization of hydrogen molecules. The location of this maximum on the electron energy scale (7.2 ev) coincides with the location of the maximum in the curve for the formation of H^- ions by the interaction of electrons and H_2O molecules.[3] Apparently maximum I owes its existence to traces of water which we were unable to remove completely from the apparatus. This assumption is supported by the fact that when hydrogen is introduced into a previously baked-out apparatus, H_2O^+ ions appear in the spectrum of positive ions. For a conclusive test, an experiment was made with water vapor in the apparatus. An intense group of H^- ions appeared, whose maximum coincided in shape and location with the one found in hydrogen. The yield curve of the H^- ions obtained from water vapor in the apparatus is shown as a dashed line in the figure.



Yield curve for H^- ions obtained by introducing hydrogen into the ion source of a mass spectrometer. Hydrogen pressure less than 2×10^{-4} mm Hg; electron current density 1.5×10^{-6} amp/mm². The absolute values of the current $i(H^-)$ are only approximate. The dashed line is the yield curve for H^- ions obtained with water vapor. The ordinates of this curve are reduced by an arbitrary factor; the scale on the ordinate axis does not refer to it.

All the facts enumerated above force one to conclude that maximum I of the hydrogen curve is to be attributed to H^- ions formed from H_2O molecules. The appearance of these ions when using hydrogen which has been subjected to elaborate purification could be explained by the formation of water molecules within the apparatus itself, as a result of the hydrogen reacting with oxide films on the metal parts of the ion source.

Maximum III and the portion IV of the curve in the figure are therfore to be attributed to H^- ions formed by the collision of electrons with H_2 molecules. Evidence for this conclusion is provided by the direct proportionality between the number of ions and the hydrogen pressure, their disappearance when hydrogen is removed from the apparatus, and also the absence of H^- ions when water vapor is bombarded with electrons in this energy range. It seems very unlikely that these ions could be formed from some other impurity in the hydrogen.

Maximum III is similar in shape to the maxima obtained when other negative atomic ions are formed by resonance capture of electrons by diatomic molecules. It is therefore natural to assume that maximum III is due to the process

$$H_2 + e \rightarrow (H_2^-) \rightarrow H^- + H. \qquad (1)$$

The difference in energy between the system $H_2 + e$ and the system $H^- + H$ can be found from the difference between the dissociation energy of a hydrogen molecule into atoms (4.4 ev) and the electron affinity of a hydrogen atom (0.75 ev), and is equal to 3.7 ev. The fact that maximum III lies at a considerably higher electron energy than this (14.5 ev) points to one of two possibilities: (1) either the H^- ion and the H atom carry away some kinetic energy (about 5 ev at the most) or (2) the H atom is formed in an excited state.

In a paper by Eyring, Hirschfelder and Taylor[4] the potential curve is calculated for the system $H + H^-$

(i.e., the H_2^- ion). The minimum in this curve lies at a considerably higher internuclear distance (about 1.8 A) than the minimum in the potential curve for the H_2 molecule (0.76 A). The internuclear separation corresponding to the fundamental vibrational level of the H_2 molecule must correspond to the steeply rising portion of the potential curve for the H_2^- ion. But if this is so, then the process of resonance capture of electrons by a hydrogen molecule requires the expenditure of a considerably greater amount of energy than the dissociation energy of the H_2^- ion, and leads to the appearance of an H^- ion with considerable kinetic energy.

The right-hand portion of the curve in the figure, denoted by the number IV, has a shape reminiscent of the yield curves of negative ions in the dissociation of a diatomic molecule into positive and negative ions. Hence we may assume that, for electron energies corresponding to region IV, the process which takes place is

$$H_2 + e \rightarrow (H_2)^* + e \rightarrow H^+ + H^- + e. \qquad (2)$$

The H_2 molecule in its ground state appears to be an atomic molecule, and dissociates into atoms upon electron bombardment. However, a series of excited states exists, as well as the ionized state.[5] If the transition to such a state is accompanied by the transfer of sufficient energy to dissociate the molecule, process (2) would be able to take place.

The dissociation energy of an H_2 molecule into H^+ and H^- ions is equal to 17.2 ev. Therefore process (2) cannot go for electron energies less than this value. In the curve shown in the figure, maximum III and the slope region IV apparently overlap. The location of the minimum between them (17 ev) is not in contradiction with this value for the energy threshold of process (2).

The curve shows still another maximum, labelled II, less pronounced than maxima I and III. Not enough data is available for us to be able to interpret this.

The apparatus used in this work did not permit the determination of an effective cross-section for the process of ion formation by the collision of electrons with gas molecules. However, comparing the yield of H^- ions with the yield of H^+ ions, and using the data of Tate and Smith[6] for the effective cross-section for ionization of H_2 molecules by electron impact, we can estimate in a very crude way the effective cross-section for process (1). At the 14.5 ev maximum, the cross-section comes out to be of the order of 10^{-21} cm^2.

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PARAMAGNETIC RELAXATION IN SINGLE CRYSTALS OF CERTAIN SALTS OF THE ELEMENTS OF THE IRON GROUP

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Measurements have been made of the constant b/c (b = constant of magnetic thermal capacity, c = Curie constant) and the spin-lattice relaxation time ρ in single crystals of certain paramagnetic salts. The dependence of b/c and ρ on the direction of the static magnetic field H relative to the crystallographic axes has been found in the copper sulfates $CuSO_4 \cdot 5H_2O$, $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ and $CuK_2(SO_4)_2 \cdot 6H_2O$. The anisotropy of b/c and ρ has not been detected, within experimental error, in single crystals of $MnSO_4 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$, $CrK(SO_4)_2 \cdot 12H_2O$.

1. INTRODUCTION

Up to the present time, investigations on paramagnetic absorption in parallel fields at room temperature have been carried out only on powders of paramagnetic salts. However, one can expect that a dependence ought to exist of the constant b/c and the spin-lattice relaxation time on the orientation of the external magnetic field H relative to the crystallographic axes.[1] The presence of such a dependence for the quantity b/c follows from the anisotropy of the static magnetic susceptibility,[2] and consequently of the transition probability A_{hk} between the energy levels of the spin system under the action of the vibrations of the lattice.[3] The transition probability A_{hk} at room temperature is determined by processes of Raman scattering of phonons.

In consequence of the fact that the energy of the phonons is on the average extremely large in comparison with the intervals between adjacent energy levels of the spin-system, the dependence of A_{hk} on the orientation of the field H should be weak. For isotropic paramagnets, the dependence of ρ on the field H is determined by the Van Vleck-Brons formula:

$$\rho = \rho_0 \frac{b/c + H^2}{b/c + pH^2}, \tag{1}$$

where $\rho \lesssim 1$, and for single crystals it will probably be the same. From all of the above, it follows that the anisotropy of the relaxation time at room temperature ought not to be large, but of the order of several per cent. However, experimental investigations that we have carried out show that the anisotropy of ρ in single crystals of $CuSO_4 \cdot 5H_2O$ is much larger.

We have carried out measurements of the absorption coefficient χ'' in single crystals of $CuSO_4 \cdot 5H_2O$, $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $CuK_2(SO_4)_2 \cdot 6H_2O$, $MnSO_4 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$ and $CrK(SO_4)_2 \cdot 12H_2O$ as a function of the magnitude and direction of the static magnetic field relative to the crystallographic axes. The constant b/c was determined by a measurement of χ'' at a frequency of 6×10^8 cps. Measurements of the paramagnetic absorption, which is due to the spin-lattice interaction, were carried out at a frequency of 10.5 megacycles. For measurement, we made use of the grid current method of Zavoiskii.[4,5] The single crystals were prepared fresh from the mother solution.

2. RESULTS OF MEASUREMENT

1. For measurement of the constant b/c, we used an Esau circuit as an oscillator with a frequency of 6×10^8 cps. The directions of the magnetic axes in the single crystals were determined in correspondence with published works.[2,6] The single crystal investigated was in the form of a sphere of diameter 8 mm.

According to Ref. 7, an individual cell in the single crystal $CuSO_4 \cdot 5H_2O$ contains two ions of Cu^{++}.

Each ion is located in a field of tetragonal symmetry, where the tetragonal axes form angles of 82° with each other. The bisectrix of the acute angle between them will be denoted by α, that of the obtuse angle by β, the normal to the $(\alpha\beta)$ plane by γ. These three directions represent the three fundamental magnetic axes.[8] In Refs. 2 and 8 it was established that the magnetic static susceptibility χ_0 and the g-factor are also isotropic for directions lying in the $(\alpha\beta)$ plane; these quantities differ markedly from their values in the $(\alpha\beta)$ plane only in the γ direction. The Tutton salts $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $CuK_2(SO_4)_2 \cdot 6H_2O$ have a structure similar to the structure of $CuSO_4 \cdot 5H_2O$.[6,8]

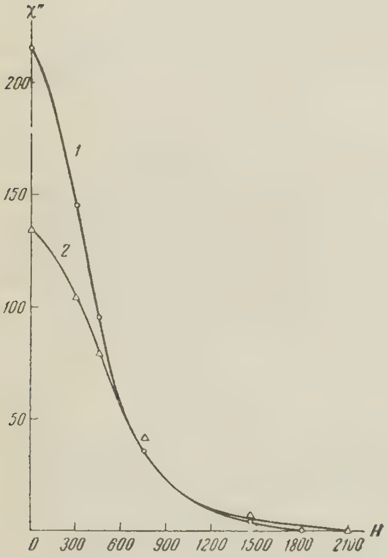

FIG. 1. Absorption curves in a single crystal of $CuSO_4 \cdot 5H_2O$ at a frequency of 6×10^8 cps as a function of the direction of the magnetic axes α and γ in relation to H.

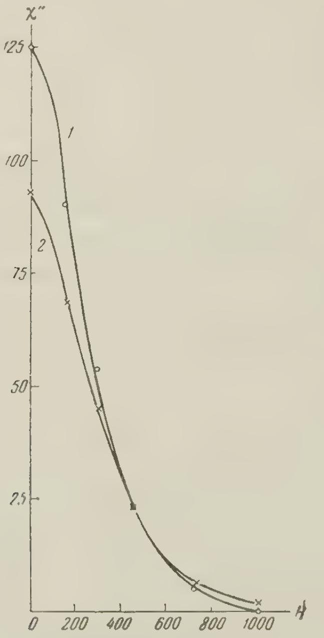

FIG. 2. Absorption curves in a single crystal of $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ at a frequency of 6×10^8 cps as a function of the direction of the magnetic axes α and γ relative to H.

Measurements of the paramagnetic absorption χ'' were carried out in their dependence of the magnitude and direction of H relative to the single crystal.* The values of χ'' for each salt differ along the axes α and β by not more than 5 — 10%. The values of χ'' along the γ axis are significantly less than the first two. We exhibit the results of measurement for the two directions: α and γ.

Results of measurement have been plotted in Figs. 1 and 2 for the salts $CuSO_4 \cdot 5H_2O$ and $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$, respectively (the curves for $CuK_2(SO_4)_2 \cdot 6H_2O$ have not been plotted). In each figure, curve 1 corresponds to the direction α, curve 2 to the direction γ. It is evident from the drawings that the zero field absorption along the γ axis is less than along the α axis; the half width δ along the γ axis is greater in comparison with the half width along the α axis. These curves are well described by Shaposhnikov's formula:[9]

$$\chi''/\chi_0 = (1 - F)^2 \rho_s \nu, \tag{2}$$

where $F = H^2/((b/c) + H^2)$, ρ_S is the isothermal spin-spin relaxation time, which does not depend on H, and ν is the frequency of the variable field.

The experimental points lie along the theoretical curve. If we assume that ρ_S does not depend on the field, then we obtain a relation for b/c from Eq. (2):[10]

$$b/c = (1 + \sqrt{2})\rho^2. \tag{3}$$

TABLE I

Material	Powder		Single Crystal			
			α axis		γ axis	
	$b/c \cdot 10^{-6}$	$\chi_0 \cdot 10^6$	$b/c \cdot 10^{-6}$	$\chi_0 \cdot 10^6$	$b/c \cdot 10^{-6}$	$\chi_0 \cdot 10^6$
$CuSO_4 \cdot 5H_2O$	0.47	6.1	0.40	6.60	0.60	5,50
$Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$	0.16	3.8	0.16	4.50	0.22	3.75
$CuK_2(SO_4)_2 \cdot 6H_2O$	0.12	3.3	0.11	3.62	0.15	3.03

The resultant values of the quantity b/c, expressed in oersteds, are given in Table I. It is seen from this table that the value of b/c for each salt is, with some approximation, inversely proportional to the magnetic static susceptibility χ_0.† Along the axis of magnetic symmetry γ, the quantity b/c is significantly larger than in the direction perpendicular to this axis. This direction lies in the $(\alpha\beta)$ plane. The results obtained for powders agree with Refs. 3 and 11. The values of b/c computed from χ'', and obtained for different oscillators frequencies of the same order, agree with each other to within 5%.

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*χ'' is expressed in relative units, H in oersteds.

†The static susceptibility χ_0 for powders was measured by the authors by the method of Gouy. The values of χ_0 for single crystals have been obtained on the basis of these results.[2,8]

Investigations of the paramagnetic relaxation in the single crystals $Mn(SO_4)_2 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$ and $CrK(SO_4)_2 \cdot 12H_2O$ give values for b/c of 6.3×10^6, 0.64×10^6, 0.28×10^6, and 0.65×10^6, respectively. These agree with the data of Gorter.[3] Here, within the limits of experimental error (5%) no dependence of b/c on the orientation of the single crystal in a magnetic field was observed.

2. The results of the measurement of the absorption coefficient in its dependence on the magnitude and direction of the external magnetic field H relative to the crystallographic axes, carried out at a frequency of 10.5 megacycles in single crystals of $CuSO_4 \cdot 5H_2O$, $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ have been plotted in Fig. 3. Curves 1 and 2 correspond to the directions α and γ for $CuSO_4 \cdot 5H_2O$, curve 3, to powders of this same salt; curves 5 and 6 correspond to the directions α and γ for $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$; curve 4, as a standard.

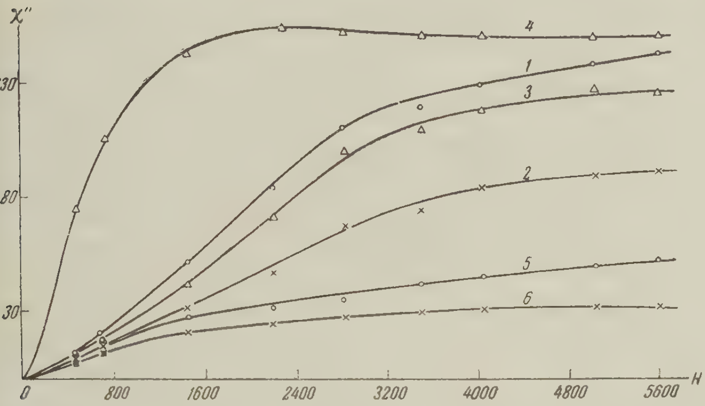

FIG. 3. Absorption curves in single crystals of $CuSO_4 \cdot 5H_2O$ and $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ at a frequency of 10.5 megacycles as a function of the direction of the magnetic axes α and γ relative to the field H.

TABLE II

Material		$CuSO_4 \cdot 5H_2O$			$Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$			$CuK_2(SO_4)_2 \cdot 6H_2O$		
Field	Powder $\rho \cdot 10^8$	α axis $\rho_\alpha \cdot 10^8$	β axis $\rho_\beta \cdot 10^8$	γ axis $\rho_\gamma \cdot 10^8$	Powder $\rho \cdot 10^8$	α axis $\rho_\alpha \cdot 10^8$	γ axis $\rho_\gamma \cdot 10^8$	Powder $\rho \cdot 10^8$	α axis $\rho_\alpha \cdot 10^8$	γ axis $\rho_\gamma \cdot 10^8$
800	1.0	1.0	1.0	0.8	0.8	0.8	0.8	0.8	0.8	0.8
1600	1.5	1.5	1.4	1.2	1.0	1.2	1.0	9.9	0.9	0.9
2400	2.0	2.5	2.3	1.7	1.2	1.3	1.1	1.0	1.0	0.9
3200	2.5	3.0	2.7	2.0	1.3	1.4	1.2	1.0	1.0	0.9
4000	2.8	3.3	3.0	2.2	1.4	1.5	1.3	1.0	1.0	0.9
4800	3.0	3.5	3.2	2.3	1.4	1.5	1.3	1.0	1.0	0.9
5600	3.1	3.6	3.3	2.4	1.4	1.5	1.3	1.0	1.0	0.9

If as seen from the drawings that the values of χ'' along the axis are much smaller than in the γ direction, for $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ and $CuK_2(SO_4)_2 \cdot 6H_2O$, these values differ by 1.4 — 1.5 times, while in the case of $CuSO_4 \cdot 5H_2O$, they differ by 1.6 — 1.7 times (in strong fields).

Measurements were carried out by a comparison method[10,11] with a single crystal of $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$ as a standard. Furthermore, a check was made on the results by comparison with the effect on a single crystal of $MnSO_4 \cdot H_2O$.

Measurements of χ'' as a function of H for the salts $MnSO_4 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$ and $CrK(SO_4)_2 \cdot 12H_2O$ show that the absorption does not depend on the orientation of the single crystal in the static field H. The experimental errors in all cases did not exceed 4%.

On the basis of the results of measurement of χ'' at a frequency of 10.5 megacycles, and of the values of b/c obtained in Sec. 1, we determined the spin-lattice relaxation times in the copper sulfates. The calculation of ρ was carried out by the formula of Casimir and Du Pre,[3] which coincides at the given frequency with the formula of Shaposhnikov:[9]

$$\chi'' = \chi_0 F m \frac{\rho \nu}{1 + \rho^2 \nu^2}. \tag{4}$$

The results of the calculation for ρ are shown in Table II.* For $CuSO_4 \cdot 5H_2O$, where a small anisotropy was observed, the results are given for the three axes α, β, γ. The data of Table II show that in the copper sulfates, the relaxation time ρ is a function of the anisotropy. The dependence of ρ on the direction of the external magnetic field H is most clearly demonstrated in the crystals of $CuSO_4 \cdot 5H_2O$. It is also of interest that where there is a large anisotropy of the relaxation time, the dependence of it on the magnitude of the applied external field H is also significant.

The results obtained for powders are described approximately by the relation $\rho = \frac{1}{3}(2\rho_\alpha + \rho_\gamma)$. A theoretical account of these results has been given by Bashkirov.[13]

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*The previously published preliminary communication[12] contained inaccuracies connected with the determination of the magnetic axes.

CONCLUSIONS

1. The constant b/c was measured as a function of the orientation of the magnetic field relative to the crystal, and is approximately inversely proportional to the static susceptibility.

2. In copper sulfates, an anisotropy was discovered in the spin-lattice relaxation time. In this case, the relaxation time is shortest when the magnetic field is directed parallel to the γ axis.

3. The anisotropy of ρ in crystals of $CuSO_4 \cdot 5H_2O$ is much greater than in the two copper sulfates $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ and $CuK_2(SO_4)_2 \cdot 6H_2O$.

4. In single crystals of $MnSO_4 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$, $CrK(SO_4)_2 \cdot 12H_2O$, no dependence of the values b/c and ρ on the orientation of the external magnetic field was observed.

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A GROUP-THEORETICAL CONSIDERATION OF THE BASIS OF RELATIVISTIC QUANTUM MECHANICS. I.

THE GENERAL PROPERTIES OF THE INHOMOGENEOUS LORENTZ GROUP

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This paper is the first of a series in which the possibilities of relativistic quantum mechanics are investigated on the basis of group theory, without the use of any specific form of the equations of motion. In the present paper we discuss some of the general properties of group representations as well as some of the specific properties of the representations of the inhomogeneous Lorentz group.

1. INTRODUCTION

THE requirements of invariance of quantum mechanics with respect to some group of transformations of space-time impose quite rigid restrictions on possible wave functions (state vectors): The wave function must transform according to one of the linear representations of the particular group. The most general group that has been treated is the inhomogeneous Lorentz group, consisting of all four-dimensional translations and rotations. A large number of unitary irreducible representations of this group were classified by Wigner.[1] However, the mathematical apparatus used by Wigner is very complicated and difficult for understanding or practical use, so that the results of this important paper have not, with rare exceptions, been used up to the present in physical investigations. In a series, of which the present paper is the first, we shall not only classify all the irreducible representations of the inhomogeneous Lorentz group, both unitary and non-unitary, but will also get them explicitly; for each of these representations we shall give the specific form of the operators for infinitesimal displacements and 4-rotations. We shall use a much simpler mathematical method than that of Wigner.

We shall also find the representations of the improper group, which includes the inversion.

The non-trivial question of time reflection will be subjected to a particular examination. The essential new feature compared to existing treatments of this problem will be the consistent use, on the one hand, of the theory of the representations of the inhomogeneous Lorentz group, and on the other, the use of the concept of a universal covering group with respect to which the spinor representations as well as the vector representations are single valued.[2] As a consequence of this procedure, we are able to obtain various new results. Thus, for example, it turns out that space-time reflections can be introduced into the universal covering group in eight non-equivalent ways. In other words, there are eight non-equivalent groups containing translations, rotations and reflections. The number of mathematically admissible non-equivalent laws of transformation under reflections for, say, the Dirac equation, turns out to be much greater than has been assumed up to now.

At present, the most detailed study available is on the homogeneous Lorentz group, which does not include displacements. The transition from the homogeneous to the inhomogeneous group is not trivial, since rotations and translations do not commute with one another. The importance of treating the inhomogeneous group is clear simply from the fact that its invariants are such fundamental quantities as mass and spin, whereas the invariants of the homogeneous group have no clear physical meaning. The fact that the representations of the homogeneous group (e.g., tensors) have wide use in physics is related to the fact that the homogeneous Lorentz group is homomorphic to the inhomogeneous group. Thus the representations of the homogeneous group are one of the classes of representations of the inhomogeneous group.

Later, by using the technique of representations developed here, we shall give the expansions of the most frequently used wave functions in irreducible components. Unlike Bargmann and Wigner,[3] we shall carry out the expansion not only for the set of solutions but for the whole domain of definition of the wave function. It turns out that contemporary theory does not use all of the unitary representations of the inhomogeneous Lorentz group, but does use a whole series of non-unitary representations. The use of the theory of representations of the inhomogeneous Lorentz group enables us to pose anew the question of possible equations for elementary particles. The unsatisfactory nature of the present scheme (cf., for example, Ref. 4), is obvious simply from the fact that it gives not a single equation which can be used for a completely correct configuration description of a single particle and which leads to positive definite normalization and energy.

The infinitesimal operators of the inhomogeneous Lorentz group are integral quantities — the total 4-momentum and total four-dimensional angular momentum, which characterize the free motion of the system as a whole. Thus, covariance with respect to this group is sufficient only for the description of free motion, and is not sufficient (though, of course, necessary) for the description of interaction. In the presence of interaction, the important quantities include, in addition to the integral quantities, densities, i.e., quantities defined at a point. In order to introduce these quantities in a covariant fashion and to establish their commutation relations, it is necessary to consider more general groups which include not only transformations of the whole space but also transformations in the neighborhood of a particular point. Questions of this sort[5] will also be treated.

The purpose of general investigations of the type presented here is to ascertain which of the specific features of contemporary theory follow by necessity from one or another of the general assumptions, which features are based on arbitrary assumptions and can be changed, and what are the limitations on these changes. In a period when the causes of the principal difficulties of the theory are not known, such investigations may prove to have more than mere academic interest.

2. DEFINITION OF THE INHOMOGENEOUS LORENTZ GROUP

A relativistic quantum equation must be invariant with respect to the transformations of the inhomogeneous Lorentz group, which includes all possible 4-rotations and 4-translations

$$x_\mu = g(x'_\mu) = a_{\mu\nu}x'_\nu + b_\mu, \tag{1}$$

where

$$a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda}. \tag{2}$$

Throughout the papers, we set $c = h = 1$; $\mu, \nu, \lambda \ldots = 1, 2, 3, 4$; $x_4 = it$. The transformations (1) include space and time reflections in addition to rotations and translations. In treating the proper inhomogeneous Lorentz group, which does not contain space and time reflections, we must impose on the coefficients $a_{\mu\nu}$ the conditions

$$\det|a_{\mu\nu}| = 1, \tag{3}$$

$$a_{44} > 0. \tag{4}$$

We shall use the symbol G to designate the proper inhomogeneous Lorentz group. If we include space reflections among the admissible transformations, but do not include time reflections (giving us the improper inhomogeneous Lorentz group, which we shall designate as G_s), then only condition (4) should be imposed on the coefficients $a_{\mu\nu}$. Finally, if we consider the full inhomogeneous Lorentz group, including reflections in both space and time, we must drop both conditions (3) and (4). We shall first treat the proper group and then generalize the results to the improper group. The question of the invariance of the equations of quantum mechanics with respect to time reflection requires special treatment.

3. IRREDUCIBLE REPRESENTATIONS OF GROUPS

Under a coordinate transformation (1), any wave function (state vector) Ω of a relativistic quantum theory must be subjected to a non-degenerate linear transformation $U(a, b)$:

$$\Omega = U(a, b)\Omega', \tag{5}$$

which depends on the coefficients $a_{\mu\nu}$, b_μ. The set of such transformations must form a representation of the group G. This means that if the successive application of two transformations g_2, g_1 of type (1) gives the transformation

$$g_2 g_1 = g_3, \tag{6}$$

then the corresponding matrices must satisfy the relation

$$U_2 U_1 = U_3. \tag{7}$$

The set of wave functions Ω can be subjected to an arbitrary linear non-degenerate transformation V

$$\Omega = V\Omega''. \tag{8}$$

When this is done, the matrix U is transformed into

$$U''(a, b) = V^{-1}U(a, b)V. \tag{9}$$

If the set of matrices $U(a, b)$ forms a representation of the group, then the set $U''(a, b)$ also gives a representation of the same group. The representations which are obtained with the matrices U, U'' are physically equivalent, and will not be considered as distinct.

It may happen that for some choice of V all the matrices U'' take on the "block" form

$$U'' = \begin{pmatrix} U^{(1)} & 0 & 0. & \ldots \\ 0 & U^{(2)} & 0 & \ldots \\ 0 & 0 & U^{(3)} & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{10}$$

where $U^{(1)}$, $U^{(2)} \ldots$ are square matrices. Then the matrices $U^{(1)}$ (and also $U^{(2)}, U^{(3)}, \ldots$) also form representations of the group G, since

$$U''(a, b)U''(a', b') = \begin{pmatrix} U^{(1)}(a, b)U^{(1)}(a', b') & 0 \ldots \\ 0 & U^{(2)}(a, b)U^{(2)}(a', b') \ldots \\ \ldots & \ldots \end{pmatrix} \tag{11}$$

If it is impossible to find a transformation V which simultaneously brings all the matrices U(a, b) to "block" form, then the representation is said to be irreducible. Representations which can be brought to the form (11), where $U^{(1)}$, $U^{(2)}$, . . . all belong to irreducible representations are said to be reducible. (For simplicity, we shall not consider representations which are not fully reducible.) It was shown by Wigner[1] that any unitary representation of the group G (i.e., a representation consisting of unitary matrices) can be expanded in irreducible representations. Therefore, at least for the unitary representations, we need only find all the irreducible representations; from them we can find all the representations of the inhomogeneous Lorentz group, i.e., all possible wave functions satisfying the requirements of relativistic invariance.

4. THE REPRESENTATIONS U^{-1T}, U^{*T}, U^{*-1}

The wave function Ω is, in general, complex and changes under a transformation of coordinates. It is essential for physical applications that there should exist real bilinear forms in Ω* and Ω which are invariant with respect to coordinate transformations. In order to prove the existence of such real bilinear invariants, we first note that if the matrices U(a, b) form a representation of the group G, then it follows from (7) that

$$U_2^{-1T} U_1^{-1T} = U_3^{-1T}, \quad \text{(I)} \qquad U_2^{*T} U_1^{*T} = U_3^{*T}, \quad \text{(II)} \qquad U_2^{*-1} U_1^{*-1} = U_3^{*-1}. \quad \text{(III)} \tag{12}$$

Here U* is the Hermitian conjugate of U, and U^T is the transpose of U. Thus the sets of matrices U^{-1T}, U^{*T}, U^{*-1} also form representations of the group G and are irreducible if the original representation U was irreducible.

In general all four representations U, U^{-1T}, U^{*T}, U^{*-1} are non-equivalent. We shall show that if the representations U and U^{*-1} are equivalent, we can construct a real bilinear invariant from Ω and Ω*.

5. THE METRIC MATRIX

If the representations U, U^{*-1} are equivalent, then by definition there exists a non-degenerate matrix h* such that

$$U(a, b) = h^{*-1} U^{*-1} h^* \tag{13}$$

for all $a_{\mu\nu}$, b_μ. From the Hermitian conjugate of (13),

$$U^* = h U^{-1} h^{-1}, \tag{14}$$

it follows that the quantity $\langle \Omega^* h \Omega \rangle$ is invariant:

$$\langle \Omega^* h \Omega \rangle = \langle \Omega'^* U^* h U \Omega' \rangle = \langle \Omega'^* h U^{-1} h^{-1} h U \Omega' \rangle = \langle \Omega'^* h \Omega' \rangle = \text{inv.} \tag{15}$$

The quantity

$$\langle \Omega^* h^* \Omega \rangle, \tag{16}$$

which is the complex conjugate of (15), will also be invariant. Adding and subtracting (15), (16), we find that the two quantities

$$J_1 = \left\langle \Omega^* \frac{h + h^*}{2} \Omega \right\rangle = \text{real}, \quad J_2 = \left\langle \Omega^* \frac{h - h^*}{2i} \Omega \right\rangle = \text{real} \tag{17}$$

are real invariants, and at least one of them is not identically equal to zero. It is not difficult to prove that the converse theorem is also true: if a non-degenerate matrix h exists, such that the quantity $\langle \Omega^* h \Omega \rangle$ is invariant, then the representations U, U^{*-1} (and also U^{-1T} and U^{*T}) are equivalent.

We can construct no more than one invariant bilinear in Ω, Ω* from functions which transform according to an irreducible representation. In fact, if we assume that in addition to the invariant (15) there exists an invariant

$$J' = \langle \Omega^* h_1 \Omega \rangle = \langle \Omega^* h \alpha \Omega \rangle,$$

where $\alpha = h^{-1} h_1$, then the relations

$$U^* h U = h, \quad U^* h \alpha U = h \alpha. \tag{18}$$

must be satisfied. Subtracting α times the first equation in (18) from the second, we get $U^*h(U\alpha - \alpha U) = 0$, or, since U^*h is non-degenerate,

$$U\alpha - \alpha U = 0. \tag{19}$$

According to Schur's Lemma, a matrix α which commutes with all the matrices of an irreducible representation can only be the unit matrix, which proves the uniqueness of the invariant for an irreducible representation. In particular, the invariants $<\Omega^* h\Omega>$ and $<\Omega^* h^* \Omega>$ must coincide, i.e., for an irreducible representation the operator h, which we shall call the "metric matrix," is Hermitian and uniquely determined to within a numerical factor.

6. REAL AND COMPLEX, UNITARY AND NON-UNITARY REPRESENTATIONS

Representations in which U and U^{*-1} are equivalent are said to be real representations. For these representations there exists a non-degenerate Hermitian matrix h such that the quantity $<\Omega^* h\Omega>$ is a real invariant. Real representations whose metric matrix has both positive and negative eigenvalues (indefinite metric) are said to be real non-unitary.

Real representations for which all the eigenvalues of the metric matrix are positive (definite metric) will be called real unitary, or simply unitary. For unitary representations, the invariant $<\Omega^* h\Omega>$ is not only real but also positive definite. In this case which is very important for quantum mechanics, the bilinear invariant possesses all the properties of the scalar product which is defined in linear algebra. By means of a suitable linear transformation, the metric matrix h can be diagonalized, i.e., brought to the form

$$(VhV^{-1})_{\alpha\beta} = h_\alpha \delta_{\alpha\beta}, \tag{20}$$

where h_α are the eigenvalues of the matrix $h_{\alpha\beta}$. For a unitary representation all the h_α are positive, and the transformation

$$\Omega = \Omega'_\alpha / \sqrt{h_\alpha}$$

reduces the diagonalized metric matrix $h_\alpha \delta_{\alpha\beta}$ to the unit matrix. The invariant then acquires the simple form

$$J = \langle \Omega'^* \Omega' \rangle. \tag{21}$$

Representations in which $U(a, b)$ is not equivalent to U^{*-1} will be called complex non-unitary, or simply complex. For such representations there exist no real (or complex) invariants bilinear in Ω^* and Ω. From (7) and (12) it follows that the product of a function Ω transforming according to (5) and a function Ω_I transforming according to U^{-1T}

$$\Omega_I = U^{-1T}\, \Omega'_I \equiv \Omega'_I\, U^{-1},$$

is invariant

$$\langle \Omega_I \Omega \rangle = \mathrm{inv} = J. \tag{22}$$

The quantity

$$J^* = \langle \Omega^* \Omega_I \rangle = \mathrm{inv}. \tag{23}$$

is also invariant. From (22), (23) it follows that the quantities

$$A = J + J^*, \quad B = i(J - J^*) \tag{24}$$

are real invariants.

We now introduce a new wave function ψ, ψ^* having twice as many components as Ω:

$$\psi = \begin{pmatrix} \Omega \\ \Omega_I^* \end{pmatrix}, \quad \psi^* = (\Omega^* \Omega_I) \tag{25}$$

and two Hermitian matrices ρ_1 and ρ_2, which act on ψ, ψ^*

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \tag{26}$$

It is obvious that the real invariants A and B are bilinear in ψ^* and ψ:

$$A = \langle\psi^*\rho_1\psi\rangle, \quad B = \langle\psi^*\rho_2\psi\rangle, \tag{27}$$

i.e. the function ψ transforms according to a real non-unitary representation. Either one of the matrices ρ_1, ρ_2 may serve as the metric matrix, as is easily verified directly.

In fact, ψ is transformed by means of the matrix

$$\begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix},$$

which satisfies condition (14) if h is chosen to be either ρ_1 or ρ_2. For example,

$$\rho_1 \begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix}^{-1} \rho_1^{-1} = \begin{pmatrix} U^* & 0 \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix}^* \tag{28}$$

Thus if the irreducible representation U(a, b) is complex, the direct sum of the representations U and U^{*-1} is a representation which is real but reducible. For convenience in writing we shall denote representations and their matrices by the same letters.

In conclusion we remark that the results of Secs. 3 — 5 are applicable to any group, since no use was made of the specific properties of the inhomogeneous Lorentz group.

7. AN EXAMPLE

As an example of a complex non-unitary representation, we may mention the two-dimensional spinor representation of the homogeneous proper Lorentz group, which consists of the set of all two-by-two unimodular matrices. The wave function (basis) of this representation is the two-dimensional relativistic spinor ψ_λ. Then the spinor χ^λ with upper index transforms according to U^{-1T}. The spinor $\chi^{\dot\lambda}$ with dotted upper index transforms according to U^{*-1}, and the spinor $\psi_{\dot\lambda}$ with dotted lower index transforms according to U^{*T}. The spinors $\psi_{\dot\lambda}$ and χ^λ transform according to non-equivalent representations, which also shows that the representation is complex. The four-component function

$$\psi = \begin{pmatrix} \psi_\lambda \\ \chi^{\dot\lambda} \end{pmatrix}$$

transforms according to a real non-unitary representation which is reducible with respect to 4-rotations, but is irreducible with respect to the improper Lorentz group which includes the inversion. The function ψ is equivalent to a Dirac wave function, and its two real invariants $\langle\psi^*\rho_1\psi\rangle$ and $\langle\psi^*\rho_2\psi\rangle$ are respectively a scalar and a pseudoscalar constructed from the Dirac matrices.

We note that the relativistic spinor representation U(a) is equivalent to the representation $U^{-1T}(a)$, which permits us to lower and raise spinor indices. This equivalence does not hold for many representations of the inhomogeneous Lorentz group, as we shall discuss when we consider the closely related question of the invariance of quantum theory with respect to time reflection.

8. THE CONDITIONS FOR RELATIVISTIC INVARIANCE OF QUANTUM THEORY

The proper group G (excluding reflections) is continuous, so that we may consider the infinitesimal 4-rotations and translations, thus considerably simplifying our investigation. Under an infinitesimal transformation (1)

$$x_\mu = x'_\mu + \varepsilon_{\mu\nu}x'_\nu + \xi_\mu\,, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \tag{29}$$

where $\epsilon_{\mu\nu}$, ξ_ν are first order infinitesimals, the wave function undergoes the transformation

$$\Omega = \left(1 + i\xi_\mu p_\mu + \frac{i}{2}\varepsilon_{\mu\nu}M_{\mu\nu}\right)\Omega'. \tag{30}$$

Here p_μ and $M_{\mu\nu}$ are operators which are independent of the coefficients $\epsilon_{\mu\nu}$, ξ_λ of the transformation.

If we demand that the transformations (30) form a representation of the group in an arbitrary infinitesimal neighborhood of the identity, we obtain the well-known commutation relations for the operators of the 4-momentum p_μ and the 4-angular momentum $M_{\mu\nu}$ (cf. for example, Ref. 6), which are the conditions of relativistic invariance of a quantum theory:

$$[M_{\mu\nu}, M_{\lambda\sigma}]_- = i(\delta_{\mu\sigma}M_{\lambda\nu} + \delta_{\mu\lambda}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\lambda} + \delta_{\lambda\nu}M_{\sigma\mu}), \quad [M_{\mu\nu}, p_\lambda]_- = i(p_\nu\delta_{\mu\lambda} - p_\mu\delta_{\nu\lambda}), \ [p_\mu, p_\nu]_- = 0. \tag{31}$$

Or, in three-dimensional form:

$$p_\mu = (\mathbf{p}, iH). \ M_{ij} = \varepsilon_{ijk}M_k, \ M_{i4} = iN_i, \tag{32}$$

$$\left.\begin{aligned} &[p_i, p_j] = 0, \ [p_i, H]_- = 0, \ [M_i, M_j]_- = -[N_i, N_j]_- = i\varepsilon_{ijk}M_k, \\ &[M_i, N_j]_- = i\varepsilon_{ijk}N_k, \ [M_i, H]_- = 0, \ [M_i, p_j]_- = i\varepsilon_{ijk}p_k, \\ &[N_i, p_j]_- = i\delta_{ij}H, \ [N_i, H]_- = ip_i. \end{aligned}\right\} \tag{33}$$

The square brackets with the minus sign (plus sign) denote the commutator (anticommutator). Latin indices take on the values 1, 2, 3; ϵ_{ijk} is the three-dimensional unit pseudotensor.

9. CONDITIONS FOR INVARIANCE WITH RESPECT TO SPACE AND TIME REFLECTIONS

The operator I_s for inversion of the space axes must commute with spatial scalars and pseudovectors, and anticommute with pseudoscalars and vectors, so that the relations

$$[I_s, p_i]_+ = 0, \ [I_s, H]_- = 0, \ [I_s, N_i]_+ = 0, \ [I_s, M_i]_- = 0. \tag{34}$$

must be satisfied.

For time reflection (without the nonlinear transformation involving a change from Ω to Ω^*), the energy H and the vector N_i change sign, while the quantities p_i and M_i remain unchanged, which leads to the relations:

$$[I_t, p_i]_- = 0, \ [I_t, M_i]_- = 0, \ [I_t, H]_+ = 0, \ [I_t, N_i]_+ = 0. \tag{35}$$

From (34) and (35) it follows that the invariant operator for reflection of all four axes, $I_{st} = I_s I_t$ satisfies the invariant commutation relations

$$[I_{st}, p_\mu]_+ = 0, \ [I_{st}, M_{\mu\nu}]_- = 0. \tag{36}$$

10. THE OPERATORS OF INTRINSIC ANGULAR MOMENTUM AND CENTER OF INERTIA

In place of the tensor $M_{\mu\nu}$ it is convenient to consider the vector g_μ:

$$g_\mu = M_{\mu\nu}p_\nu \tag{37}$$

and the pseudovector Γ_σ:

$$\Gamma_\sigma = (1/2i)\,\varepsilon_{\sigma\mu\nu\lambda}M_{\mu\nu}p_\lambda, \tag{38}$$

which satisfy the identities

$$g_\mu p_\mu \equiv 0, \ \Gamma_\sigma p_\sigma \equiv 0. \tag{39}$$

In (38), $\epsilon_{\sigma\mu\nu\lambda}$ is the completely antisymmetric unit pseudotensor. Geometrically, the pseudovector Γ_σ gives the 4-rotations in the plane perpendicular to p_μ, while the vector g_μ gives the remaining 4-rotations. In Ref. 7 it was shown that g_μ is the operator of the center of inertia, while Γ_σ is the intrinsic angular momentum of the isolated quantum mechanical system.

From (37), (38) we find that

$$M_{\mu\nu} = (g_\mu p_\nu - g_\nu p_\mu + i\varepsilon_{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda)(p_\xi^2)^{-1}, \tag{40}$$

i.e., $M_{\mu\nu}$ and the pair of quantities g_μ, Γ_σ, determine one another uniquely. Expression (40) becomes meaningless for $p_\xi^2 = 0$, so the case of zero rest mass will be treated separately.

From (31), (37), (38), we can get the commutation relations

$$\left.\begin{aligned} &[p_\mu, \Gamma_\sigma]_- = 0, \ [\Gamma_\sigma, \Gamma_\xi]_- = p_\lambda\Gamma_\mu\varepsilon_{\mu\sigma\xi\lambda}, \ [p_\mu, p_\nu]_- = 0, \\ &[g_\mu, \Gamma_\sigma]_- = -i\Gamma_\mu p_\sigma, \ [g_\mu, p_\nu]_- = -i(p_\mu p_\nu - \delta_{\mu\nu}p_\lambda^2), \\ &[g_\mu g_\nu]_- = -i(g_\mu p_\nu - g_\nu p_\mu + i\varepsilon_{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda) = -iM_{\mu\nu}p_\lambda^2 \end{aligned}\right\} \tag{41}$$

$$[I_s, p_i]_+ = 0, \ [I_s, p_0]_- = 0, \ [I_s, \Gamma_i]_- = 0, \ [I_s, \Gamma_4]_+ = 0, \ [I_s, g_i]_+ = 0, \quad [I_s, g_4]_- = 0, \tag{42}$$

$$[I_{st}, p_\mu]_+ = 0, \ [I_{st}, \Gamma_\sigma]_+ = 0, \ [I_{st}, g_\mu]_+ = 0. \tag{43}$$

11. THE INFINITESIMAL GROUP AND THE GROUP IN THE LARGE

Every irreducible representation of a continuous group will also be a representation in the neighborhood of the identity. The converse statement is not true in general. Thus, if we find all the irreducible representations of the group G in the neighborhood of the identity, i.e., if we find operators $M_{\mu\nu}$, p_λ satisfying (31), then we have not overlooked any of the irreducible representations of the whole group. However, representations may occur which are continuous only in the neighborhood of the identity, but not over the whole group. Such representations must be found and discarded.

When the improper rotations are included in the group, we must include the commutation relations of the operators $M_{\mu\nu}$, p_λ, or the set p_μ, g_ν, Γ_σ with the reflection operators.

12. INVARIANTS OF A GROUP AND THEIR CONNECTION WITH IRREDUCIBLE REPRESENTATIONS

From the well-known Schur lemma[2] in the theory of group representations, it follows that the necessary and sufficient condition for a representation to be irreducible is that the only operator which commutes with all the matrices of the representation is the unit operator. From this it follows that if the group contains an operator which commutes with all the elements of the group,* then the representation can be irreducible only if all the functions which appear in it belong to the same eigenvalue of this operator. In fact, if this is not the case, the representation will be reducible, since it will contain an operator different from the unit operator and commuting with all the elements of the group.

On the other hand, if we find all the independent invariants of a group and form a representation all of whose functions are eigenfunctions belonging to the same eigenvalue of each of the invariants of the group, the representation will be irreducible, since each of the invariants is the unit operator in this representation and, by definition, there are no other operators which commute with all the elements of the group. In other words, to each complete set of eigenfvalues of all the invariants of the group there corresponds one and only one irreducible representation. Thus the problem of classifying the irreducible representations of a group reduces to finding the eigenvalue spectra of the invariants of the group.

In seeking the invariants of the group, we note first that only a scalar can commute with the operator $M_{\mu\nu}$. (Here a scalar is a quantity which is invariant under the transformations of the homogeneous Lorentz group.) For example, for any vector operator A,

$$[M_{\mu\nu}A_\lambda]_- = i\,(A_\nu\delta_{\mu\lambda} - A_\mu\delta_{\nu\lambda}). \tag{44}$$

As a consequence of (39), (40), there are four independent scalars in the proper inhomogeneous Lorentz group:

$$p_\mu^2,\ \Gamma_\sigma^2,\ g_\mu^2,\ g_\mu\Gamma_\mu. \tag{45}$$

All others scalars, for example, $M^2{}_{\mu\nu}$, $M_{\mu\nu}M_{\nu\lambda}M_{\lambda\mu}$, $M_{\mu\nu}M_{\nu\lambda}M_{\lambda\sigma}p_\mu p_\sigma$, etc. can be expressed in terms of the quantities (45) by using (40) and (41).

Of the four scalars in (45), only two, p_μ^2 and Γ_σ^2, commute not only with $M_{\mu\nu}$ but also with p_λ, i.e., are invariants of the group G. The scalars g_μ^2 and $g_\mu\Gamma_\mu$, and any combination of them, do not commute with p_μ and are not invariants of the group G.

We emphasize that the quantities $M_{\mu\nu}^2$ and $M_{\mu\nu}M_{\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma}$, which are the invariants of the homogeneous Lorentz group (not including translations), do not commute with p_λ and are therefore not suitable for classification of the states of relativistic quantum mechanical systems. Other invariants in addition p_μ^2 and Γ_σ^2 are possible for particular classes of representations. Thus, relation (41) is not contradicted by the equality

$$\Gamma_\sigma = \Sigma p_\sigma, \tag{46}$$

where Σ is a number. We then get from (39),

$$p_\sigma^2 = 0. \tag{47}$$

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*In mathematics, such operators are said to be central. The use of the term invariant is more appropriate for physics, since any central operator which has a physical meaning always refers to an invariant physical quantity.

Thus if the first invariant (the square of the mass) is zero, we can have proportionality between the vectors Γ_σ and p_σ, and the proportionality factor is an invariant. (We remind the reader that we have not yet considered reflections, so that we do not distinguish between scalars and pseudoscalars.) For time-like (c-number) 4-vectors, the sign of the fourth component is an invariant. For certain representations of the group G, the signs of the fourth components of p_μ and also of Γ_σ turn out to be invariant operators. Finally, the relations (31) are not contradicted by the equality

$$p_\mu = 0. \tag{48}$$

For these representations the invariants are the quantities $M^2_{\mu\nu}$ and $M_{\mu\nu}M_{\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma}$, which are the invariants of the homogeneous Lorentz group.

13. THE CONDITION FOR REALITY OF THE EIGENVALUES OF THE INVARIANTS

The purpose of the present section is to prove the following theorem:

An irreducible representation of the group G is real in the sense of Sec. 6 if and only if the eigenvalues of all the invariants of the group are real for this representation. For real irreducible representations, the eigenvalues of all the components of the operators $M_{\mu\nu}$, p_λ, Γ_σ are also real.

By definition, the condition for reality of a representation is the equivalence of the representations

$$U(a, b) \text{ and } U^{-1^*}(a, b).$$

The infinitesimal transformation of type (30) for function Ω_{III}, which transforms according to the representation U^{*-1}, has the form

$$\Omega_{III} = (1 + i\xi_\mu p^*_\mu + \frac{i}{2}\varepsilon_{\mu\nu}M^*_{\mu\nu})\,\Omega'_{III}. \tag{49}$$

We mention that by the vector p^*_μ we mean the vector with components $(\mathbf{p}^*, ip^*_0)$, and by the tensor $M^*_{\mu\nu}$ we mean the tensor with components $(\mathbf{M}^*, i\mathbf{N}^*)$. This point has no basic significance. The use of covariant and contravariant components with the usual definition of the adjoint will give the same results. From (39) and (40) it follows that the representations U, U^{*-1} are equivalent if the operators $M_{\mu\nu}$, p_λ are equivalent to the operators $M_{\mu\nu}$, p_λ. By taking the Hermitian conjugate of (31), we find that the operators $M_{\mu\nu}$, p_λ always form a representation of the group G, since they satisfy the same commutation relations as the original operators $M_{\mu\nu}$, p_λ. The invariants of this new representation are the complex conjugates of the invariants of the original representation. Thus if any of the eigenvalues of the invariants are complex for the irreducible representation U, the representations U and U^{*-1} are equivalent. But if all the eigenvalues of the invariants are real for the irreducible representation U, then U is equivalent U^{*-1}.

If any one of the components $M_{\mu\nu}$, p_λ is equivalent to its adjoint, the two must coincide when brought to diagonal form, i.e., they must be real. This completes the proof of the theorem. We note that the proof makes essential use of the irreducibility of the representation U. Thus, real irreducible representations are characterized by real eigenvalues of the invariants and the operators of infinitesimal transformations.

According to Sec. 5, for irreducible representations there exists a Hermitian metric matrix h such that $\langle\Omega^* h\Omega\rangle = \text{inv}$. From the invariance of this quantity with respect to infinitesimal transformations, it follows that

$$hM_{\mu\nu} - M^*_{\mu\nu}h = 0,\ \ hp_\lambda - p^*_\lambda h = 0. \tag{50}$$

From (50), we easily find that

$$h\Gamma_\sigma - \Gamma^*_\sigma h = 0, \tag{51}$$

i.e., the operators $hM_{\mu\nu}$, hp_λ, and $h\Gamma_\sigma$ are Hermitian. For unitary representations, h = 1 and the operators $M_{\mu\nu}$, p_λ, and Γ_σ are Hermitian. In the general case, the transformation matrices are non-unitary and the operators $M_{\mu\nu}$, p_λ and Γ_σ are, in general, non-Hermitian even if their eigenvalues are real.

For a real irreducible representation which is chosen in the form where all the operators of some complete set are diagonal (for example, p_1, p_2, p_3, p_0, Γ^2_σ, Γ_3), the relations (50) and (51) become

$$[p_\lambda, h]_- = 0,\ [\Gamma_\sigma^2, h]_- = 0,\ [\Gamma_3, h]_- = 0. \tag{52}$$

Then the metric matrix h, which commutes with all the operators of the complete set, must also be diagonal.

It may appear strange that, for example, the operators p_λ and h commute in one but not in another of two equivalent representations. This occurs because the transformation for h is different from that for all the other operators:

$$\Omega = V\Omega',\ \Omega^* = \Omega'^* V^*, \qquad \langle \Omega^* h p_\lambda \Omega \rangle = \langle \Omega'^* V^* h V V^{-1} p_\lambda V \Omega' \rangle = \langle \Omega'^* h' p'_\lambda \Omega \rangle; \qquad h' = V^* h V;\ p'_\lambda = V^{-1} p_\lambda V. \tag{53}$$

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ELECTRICAL CONDUCTIVITY OF FERROMAGNETIC SEMICONDUCTORS (FERRITES)

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A ferrite is considered as a lattice of classical magnetic dipoles submerged in a dielectric continuum. An analysis of the electron conductivity of such a model shows that the line ln $\lambda \sim T^{-1}$ (λ is the electrical conductivity) must have a break at the Curie point, in agreement with experiment, as the activation energy in the ferromagnetic region decreases.

KOMAR and Kliushin[1] detected a break in the line $\ln \lambda = f(T^{-1})$ (λ is electrical conductivity) for ferrites in the transition through the Curie point, where the activation energy in the ferromagnetic region is less than in the paramagnetic region. The fact that the break is observed at precisely the Curie point indicates a connection between this phenomenon and the presence of spontaneous magnetization. We show that the existence of this break finds a simple explanation on the basis of a theory that takes into account the interaction of the conduction electrons and the electrons of the unoccupied shells of the magnetic ions.[2]

Ferrites have an electron conductivity due to the stoichiometric excess of metal[3] or to the presence of impurities.[4] The problem of electron motion in a lattice of nonmagnetic ionic crystals has already been solved under the assumption that the ion lattice can be replaced by a dielectric continuum (polaron theory, Refs. 5 and 6). It is natural to use this method for ferrites, which essentially are also ionic crystals.

In order to take their magnetic properties into account, let us assume that spins, located in a certain spatial lattice so that they can be separated into two magnetic sublattices, are "impregnated" in the continuum. The sublattice magnetizations are anti-parallel and cancel each other incompletely. As in the Néel[7] theory, let us assume that there is only one kind of magnetic ion, i.e., one kind of spin.

The Hamiltonian of a ferrite containing one "excess" conduction electron is

$$H = p^2/2m + H_l + H_i, \tag{1}$$

where H_ℓ is the lattice Hamiltonian and H_i the Hamiltonian of electron interaction with the lattice.

To the accuracy of terms of the same order as the anharmonic terms, H_ℓ is composed additively of the Hamiltonian of the optical vibrations H_κ and of the mutual spin energy of the lattice H_c. Acoustic vibrations cannot be taken into account since it is known that the electrons in ionic crystals are scattered principally by optical vibrations. In nonmagnetic ionic crystals, H_i will be composed of the periodic potential $V_0(\mathbf{r})$ and of the interaction with the inertial polarization $H_{i\kappa}$ (Refs. 5 and 6) ($\mathbf{r}$ is the electron coordinate). Interaction of the electron with the spin of the system is added thereto in ferrites.

Choosing the separated magnetic axis as the z axis, let us assume as the periodic part of this interaction

$$U(\mathbf{r}) = 2s_z V_1, \quad V_1 = \eta_1 W(\mathbf{r}) - \eta_2 W(\mathbf{r} - \mathbf{h}), \tag{2}$$

where s_z is the z component of the electron spin operator, η_1 and η_2 are the absolute values of the relative magnetizations of the first and second sublattices; $W(\mathbf{r})$ is a periodic function.

The Hamiltonian (1) becomes

$$H = p^2/2m + V_0(\mathbf{r}) + 2s_z V_1(\mathbf{r}) + H_\varkappa + H_{i\varkappa}. \tag{3}$$

H_c can be discarded as a constant. The explicit forms of H_κ and $H_{i\kappa}$ are not required. The expression (3) differs from the Hamiltonian of the polaron problem only in the addition of $2s_z V_1(\mathbf{r})$ to the electron periodic potential.

Let us be limited to the case of weak electron coupling to the polarization vibrations and, consequently let us consider $H_{i\kappa}$ as a perturbation. Separating the optical variables, we arrive at the problem of an electron in a periodic field with the Hamiltonian

$$H_e = p^2/2m + V_0(\mathbf{r}) + 2\sigma V_1. \tag{4}$$

The spin variable is evidently separated out, which would lead to the replacement of s_z by the electron spin quantum number σ ($\sigma = \pm 1/2$). Let us consider the electron energy spectrum. In order for the crystal to have ferrite properties, i.e., for it to have non-equivalent magnetic sublattices, it must be assumed that the sites $\mathbf{R}$ and $\mathbf{R} + \mathbf{h}$ of the first and second sublattices are crystallographically non-equivalent. (For example, this is attained in the spinel lattice because of the dissimilar coordinations of these sites by the oxygen ions). The non-equivalence is retained even if the added term $2\sigma V_1$ is neglected. Consequently, V_0 and V_1 have the same periodicity. Therefore, the energy spectrum will always have a band structure. The presence of the added term only introduces a small quantitative increment.* Actually, the estimate assumed for V_1 in Ref. 2 for metals can be retained: $V_1 \sim 0.2 - 0.5$ ev. The width of the permitted band ΔE is of the order of several ev. Consequently, the increments will be of the order of $V_1/\Delta E$. Let us expand the electron energy in this quantity:

$$E_\sigma(\mathrm{K}) = E^{(0)}(\mathrm{K}) + 2\sigma E^{(1)}(\mathrm{K}) + \ldots \tag{5}$$

(K is the quasi-wave vector). Using also the effective-mass method, we finally obtain

$$E_\sigma(\mathrm{K}) = 2\sigma\varepsilon + \hbar^2 K^2/2\mu, \quad \mu = \hbar^2/(d^2E^{(0)}/dK^2)_0, \tag{6}$$

---

*The sublattice sites in antiferromagnetics must be crystallographically equivalent so that their magnetizations cancel. The magnetic ions located therein will create identical force fields for an electron if H_{ic} is not taken into account. The force fields of the magnetic ions of different sublattices are different when H_{ic} is taken into account. Consequently, $V_0 + 2\sigma V_1$ will have "twice" the period of V_0. This leads to the splitting of each of the permitted bands into two with a transition from the paramagnetic region into a region below the Curie point, as Slater[8] already has remarked.

where μ is the effective mass; $\epsilon = E^{(1)}(0)$. For simplicity, it is assumed that $K = 0$ corresponds to the minimum energy in the conduction band and that the effective mass is isotropic; the energy is measured from the position of the bottom of the conduction band in the paramagnetic region $(V_1 = 0)$, in this connection, the constant $E^{(0)}(0)$ is discarded; the term proportional to $d^2E^{(1)}/dK^2$ will be of higher order since $h^2K^2/2\mu\Delta E$ is also small. Without loss of generality, it can be considered that $\epsilon > 0$. In the opposite case, the parts of the "right" and "left" conduction electrons must be interchanged.

As is seen from (6), the bottom of the "right" electron band is raised by $\epsilon \sim 0.2$ ev and the "left" is lowered by the same quantity when the transition is made from the paramagnetic into the ferromagnetic region.

It should be noted that we at once obtain the energy spectrum (6) in the Sommerfeld approximation, where

$$V_0 \to \Omega_0^{-1}\int V_0 d\mathrm{r}, \quad V_1 \to \Omega_0^{-1}\int V_1 d\mathrm{r}, \quad \varepsilon = (\eta_1 - \eta_2)\,\Omega_0^{-1}\int W d\mathrm{r},$$

and μ is replaced by the actual mass m (the integrals are taken over the volume Ω_0 of the elementary cell).

In order to determine the chemical potential of the electron gas ζ, let us assume that the conductivity is caused by a univalent metal impurity. We will have the donor level at a distance $\Delta E'$ below the bottom of the conduction band. The intrinsic conductivity can be considered insignificant. The potential ζ is determined according to Ref. 9 from the equation (n_0 is donor concentration)

$$n_0 = n_0\left[\exp\frac{-\Delta E' - \zeta}{kT} + 1\right]^{-1} + \frac{1}{2\pi^2}\int \exp\left(-\frac{\hbar^2K^2}{2\mu kT}\right)K^2dK\sum_{\sigma}\exp\left(\frac{\zeta - 2\sigma\varepsilon}{kT}\right),$$

from which we obtain

$$\exp\frac{\zeta}{kT} = n_0^{1/2}(2\hbar)^{3/2}\left(\frac{\pi}{2\mu kT}\right)^{3/4}\left(\sum_{\sigma}\exp\frac{2\sigma\varepsilon}{kT}\right)^{-1/2}\exp\left(-\frac{\Delta E'}{2kT}\right).$$

For the ferromagnetic region $\epsilon \gg kT$ and, consequently, $\exp(-\epsilon/kT)$ can be neglected in comparison with $\exp(\epsilon/kT)$. We then obtain for the electron concentration $n = n_r + n_\ell$ (the subscripts r and ℓ refer to "right" and "left" electrons, respectively):

$$n = n_r + n_l = \sqrt{2}\,n_0^{1/2}(2\hbar)^{-3/2}(2\mu kT/\pi)^{3/4}\exp(-\Delta E'/2kT) \tag{7}$$

in the paramagnetic region $(\epsilon = 0)$,

$$n = n_l = n_0^{1/2}(2\hbar)^{-3/2}\left(\frac{2\mu kT}{\pi}\right)^{3/4}\exp\left(-\frac{\Delta E' - \varepsilon}{2kT}\right), \quad n_r \approx 0 \tag{8}$$

below the Curie point. The electron mobilities above and below the Curie point will be approximately identical since we do not take the scattering caused by the term $(H_{ic} - U)$ into account.

We see from (7) and (8) that the transition from the paramagnetic into the ferromagnetic region will be accompanied by a break* in the line $\ln\lambda \sim T^{-1}$, where the slope in the second case will be smaller (independently of the sign of ϵ). Hence, the reason for the break is the lowering of the bottom of the band for conduction electrons of one of the two spin orientations. Experimental values of the activation energy usually do not exceed 0.5 ev. The slope is halved, roughly, for $\epsilon \sim 0.2$ ev, in full agreement with experiment.

The theory leads to a curious consequence. We obtain for the ratio of the concentrations n_r and n_ℓ

$$(n_r/n_l) = \exp(-2\varepsilon/kT),$$

i.e., electrons of just one spin orientation will exist almost exclusively in the conduction band below the Curie point.

For antiferromagnetics $(\eta_1 = \eta_2)$ $\epsilon = 0$, i.e., if the reason cited for the break is unique, the break should be absent in this case, at least, for the model chosen.

In conclusion, I take this opportunity to express my thanks to Prof. E. I. Kondorskii for reviewing the work and for discussions.

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*The mobilities in ionic crystals depend exponentially on $\hbar\omega_0/kT$, where ω_0 is the limiting frequency. This factor can be considered included in $\exp(-\Delta E'/2kT)$.

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PROPAGATION OF CASCADES IN A MULTI-LAYER MEDIUM

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The propagation of a cascade consisting of several types of particles in a medium composed of homogeneous layers R^λ is considered. The layer boundaries may be mobile. The cascade particles collide with the particles of the medium and in the process are absorbed, scattered, and produce new particles. The functions W^λ describing the distribution of particles of each type according to position and velocity are found under the assumption that the particle distribution functions V^λ in the layer R^λ when the layer occupies the whole space are known. The functions W^λ are represented as infinite series of integrals which, in general, show a good convergence. The integrands are certain products of the functions V^λ.

In the present article we shall make use of the notation and results of Ref. 1, in which a method of calculating the functions $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})$ was given. The functions V_{ij} determine the probabilities $V_{ij}\,d\mathbf{r}\,d\mathbf{v}$ that a particle of a given type A_j will be found at the time t to possess radius vector between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ and velocity between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ if the cascade is initiated by a single particle appearing with the velocity $\mathbf{u}$ at the time s and the point $\mathbf{q}$. It has been assumed that the medium is filling the whole space and that its properties are independent of time and place. In Ref. 1 it has been assumed that all new particles of the cascade are produced at the moment of collision. In Ref. 2, an analogous function V describing a neutron cascade — a cascade consisting of particles of a single type which may be produced with a delay — was found by means of the same method. It is easy to combine the two cases and to consider a cascade consisting of n types of particles which may be produced with a delay. In that case, too, we shall assume the functions V_{ij} to be known for the case of an infinite homogeneous medium. In the present, a method of solution of this problem will be given for the case of a multi-layer medium, i.e., a medium consisting of different homogeneous layers occupying adjoining regions $R^\lambda (\lambda = 1, 2 \ldots)$. Let E be the space of variables $t, \mathbf{r}, \mathbf{v}, j$. It consists of n 7-dimensional spaces $E_j (j = 1, 2 \ldots n)$. We shall assume that the boundaries are varying with time t. Furthermore, for the sake of symmetry, we shall assume that they may depend also on j and $\mathbf{v}$, i.e, that the R^λ are arbitrary regions in E.

The solution of this problem is important, for example, for the study of transient cosmic ray effects,[3,4] in the theory of nuclear reactors,[5] in calculations of radiation shielding,[6] etc. The problem amounts to solving the Boltzmann linear integral equation of the type (1.15) or (1.16)* in a multi-layer medium and represents a generalization of an analogous problem in the theory of parabolic differential equations.[7,8]

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*Here and in the following (1.15) denotes the formula (15) of Ref. 1, etc.

If we put $P = (s, \mathbf{q}, \mathbf{u}, i)$ and $Q = (t, \mathbf{r}, \mathbf{v}, j)$ then the functions $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})$ can be written in an abbreviated form as $V(P, Q)$. We shall denote the function $V(P, Q)$ by $V^{\lambda}(P, Q)$ if the whole space is filled by the medium filling the region R^{λ}. We shall assume the latter functions to be given. We shall denote by $W(P, Q)$ the required function describing the cascade particle distribution in a multi-layer medium, and by $W^{\lambda}(P, Q)$ the expression of this function in the region R^{λ}. We shall also introduce the auxiliary functions $U^{\lambda}(P, Q)$. The function U^{λ} gives the particle distribution in the region R^{λ} if all other regions are filled with an ideally absorbing medium, so that any particle leaving R^{λ} disappears immediately. Evidently, $U^{\lambda}(P, Q) = 0$ if P lies within the region R^{λ}.

Before calculating the function W^{λ} we shall solve an auxilliary problem. The functions $V(P, Q)$ give the particle distribution at the time t according to particle type A, position $\mathbf{r}$ and velocity $\mathbf{v}$. It may be, however, required to solve an inverse problem of the particle distribution for a given value of a certain component of $\mathbf{r}$ or $\mathbf{v}$ according to the type A_j, time t, and the remaining components of $\mathbf{r}$ and $\mathbf{v}$. In general, we can make the following substitution for the variables t, $\mathbf{r}$, $\mathbf{v}$, j in E:

$$t = t_j(\tau, \sigma), \quad \mathbf{r} = \mathbf{r}_j(\tau, \sigma), \quad \mathbf{v} = \mathbf{v}_j(\tau, \sigma), \tag{1}$$

where σ is a 6-component variable, so that τ and σ are seven new variables and putting $P^* = (\tau, \sigma, j)$ we find the probability $V^*(P, P^*)\,d\sigma$ of finding a particle of the type A_j with σ in the interval between σ and $\sigma + d\sigma$ for given values of j and τ. Taking into account that the variable j varies in a discrete way we can see that, without any loss of generality, we have preserved its value carrying out the transformation (1).

Let K_0 be the original particle of the cascade and K one of the cascade particles existing at the time t. It has been shown[1] that corresponding to the latter particle is a chain C of particles connecting K_0 and K. All points Q in E which correspond to the particles of the chain C at the time intervals between the corresponding collisions, form a certain curve C which will be called the cascade development curve. The points in E corresponding to collisions divide this line into a finite number of arcs B, the velocity $\mathbf{v}$ varying continuously and j remaining constant along each arc. Every arc lies completely in one of the spaces E_j. By virtue of (1.1):

$$dt = \frac{d\mathbf{r}}{\mathbf{v}} = \frac{d\mathbf{v}}{\mathbf{F}_j(t, \mathbf{r}, \mathbf{v})};$$

clearly, a single development curve of a completely determined direction passes through every point of the spaces E and E_j Let $\mathbf{N}_j$ be a unit vector in E_j determining this direction. Let $\mathbf{n}_j$ and $\boldsymbol{\nu}_j$ be unit vectors normal to the surfaces $t = \text{const}$ and $\tau = \text{const}$ in E_j. It is evident that

$$\mathbf{N}_j = (1, \mathbf{v}, \mathbf{F}_j) \Big/ \sqrt{1 + v^2 + F_j^2}, \quad \mathbf{n}_j = (1, 0, 0)$$

and that the components of the vector $\boldsymbol{\nu}_j$ are given by the sixth-order determinants obtained from the matrix

$$\left(\frac{\partial t_j}{\partial \sigma}, \frac{\partial \mathbf{r}_j}{\partial \sigma}, \frac{\partial \mathbf{v}_j}{\partial \sigma}\right),$$

(which has 6 rows and 7 columns), divided by the square root Δ_j of the sum of their squares. Taking it into account that the mean length of the arcs B is always finite in E_j and that at least in the vicinity of the points of intersection of the surfaces $t = \text{const}$ and $\tau = \text{const}$ every development line passes through both surfaces, we obtain $V/\mathbf{N}_j\mathbf{n}_j = V/\mathbf{N}_j\boldsymbol{\nu}_j\Delta_j$ or, substituting for $\mathbf{N}_j$, $\mathbf{n}_j$, $\boldsymbol{\nu}_j$ their values given above, we obtain the following expression for the required probability:

$$V^*(P, P^*) = V(P, Q) \begin{vmatrix} 1 & \mathbf{v} & \mathbf{F}_j(t, \mathbf{r}, \mathbf{v}) \\ \frac{\partial t_j}{\partial \sigma} & \frac{\partial \mathbf{r}_j}{\partial \sigma} & \frac{\partial \mathbf{v}_j}{\partial \sigma} \end{vmatrix}, \tag{2}$$

where all the variables t, $\mathbf{r}$, $\mathbf{v}$ on the right-hand side of the equation should be expressed by τ and σ according to (1) (i and j do not change). If, for example, we want to find the probability that one particle of the type A_j traverses the area between x, y, and $x + dx$, $y + dy$ on the surface $z = \text{const}$ with a velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ within the time interval between t and $t + dt$ then, according to Eq. (2) we have:

$$V_{ij}^*(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\,dt\,dx\,dy\,d\mathbf{v} = V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\,v_z\,dt\,dx\,dy\,d\mathbf{v}. \tag{3}$$

Since the derivatives of the functions (1) with respect to τ do not enter into Eq. (2), it is clear that in order to find the value of the function V^* on a given surface $\tau = \text{const}$ it is necessary to know the functions (1) for that value of τ only.

Let R^μ and R^ν be two adjoining regions in E and let $S^{\mu\nu}$ be their common boundary surface, decomposing into surfaces $S_j^{\mu\nu}$ in the spaces E_j. Let $P^* = (\tau, \rho, \nu, k)$ be a point on the surface $S^{\mu\nu}$, and $F^\mu(P)$ and $G^\nu(P)$ be two functions of P defined in R^μ and R^ν respectively. We shall introduce the following notation:

$$F^\mu G^\nu = \sum_k \int\limits_{S_k^{\mu\nu}} F^\mu(P^*) \begin{vmatrix} 1 & \mathbf{r} & \mathbf{F}_k(\tau, \rho, \mathbf{v}) \\ \frac{\partial\tau}{\partial\sigma_k} & \frac{\partial\rho}{\partial\sigma_k} & \frac{\partial \mathbf{v}}{\partial\sigma_k} \end{vmatrix} G^\nu(P^*)\, d\sigma_k, \tag{4}$$

where the integration is to be carried out over only those parts of the surfaces $S_j^{\mu\nu}$ where the vectors N_j, determining the direction of the development curves in E_j, are oriented from R^μ into R^ν. If the function $F^\mu(P)$ gives the distribution of cascade particles in the region R^μ and $G^\nu(P) = U^\lambda(P, Q)$ then, according to Eq. (2) and to the definition of the function U^λ, it is evident that the product $F^\mu U^\nu$ [in the sense of (4)] gives the distribution of cascade particles in the region R^ν, which are those of the particles from R^μ which passed into R^ν and the development curves of which lie totally in R^ν.

In the case when the surface $S^{\mu\nu}$ is the plane $z = z_0$, we can put simply $\sigma = (t, x, y, v)$, and from Eq. (4) we obtain

$$F^\mu U_j^\nu(t, \mathbf{r}, \mathbf{v}) = \sum_k \int\limits_{-\infty}^{t} \iiiint\!\!\int\limits_{0}^{\infty} F_k^\mu(\tau, \rho, \boldsymbol{\nu})\, \nu_z\, U_{kj}^\nu(\tau, \rho, \boldsymbol{\nu}, t, \mathbf{r}, v)\, d\tau\, d\xi\, d\eta\, d\nu_x\, d\nu_y\, d\nu_z \quad \text{for } \rho = (\xi, \eta, z_0). \tag{5}$$

We shall now pass to the solution of the main problem. Let a cascade originate in the point P of the region R^κ. We shall say that a cascade particle, present at the time t and represented in E by a point Q, is of the m-th order when its development curve, going from P to Q, passes through m regions R^λ. If the curve traverses a certain region several times, the region contributes a corresponding number of times to the order of the particle. Let $W_m^\lambda(P, Q)$ denote the required function $W^\lambda(P, Q)$ under the complementary condition that only particles of the m-th order are considered. We have then

$$W^\lambda(P, Q) = \sum_{m=1}^{\infty} W_m^\lambda(P, Q). \tag{6}$$

But it is evident that

$$W_1^\lambda = \begin{cases} U^\kappa(P, Q) & \text{for } \lambda = \kappa, \\ 0 & \text{for } \lambda \neq \kappa, \end{cases} \tag{7}$$

and taking (4) into account, we obtain

$$W_{m+1}^\lambda(P, Q) = \sum_\mu W_m^\mu(P, P^*)\, U^\lambda(P^*, P) \quad (m = 1, 2, \ldots). \tag{8}$$

Equations (6), (7), and (8) make it possible to calculate W^λ when U^λ is known. The functions W^λ are therefore obtained in the form of an infinite series of products of the type (4), i.e., integrals, the integrands of which are certain products of the functions U^λ. It should be noted that the sum (8) is to be extended only over those values of μ to which correspond regions R^μ adjoining to R^λ.

If there are, for example, only two regions R^0 and R' in the whole space and if the point P lies within R^0, we have

$$W_1^0 = U^0(P, Q),\ W_1' = 0, \quad W_{m+1}^0 = \begin{cases} 0 \\ W_m' U^0, \end{cases} \quad W_{m+1}' = \begin{cases} W_m^0 U' & \text{for odd } m \\ 0 & \text{for even } m \end{cases}$$

and, consequently, we obtain the following infinite series for W^0 and W':

$$W^0 = U^0 + U^0U'U^0 + U^0U'U^0U'U^0 + \ldots, \quad W' = U^0U' + U^0U'U^0U' + U^0U'U^0U'U^0U' + \ldots$$

These yield immediately the integral equations for W^0 and W':

$$W^0 = U^0 + U^0U'W^0, \ W' = U^0U' + U^0U'W'. \tag{9}$$

In order to find the functions U^λ we shall denote by $\overline{R}^\lambda$ the region containing the whole space E outside R^λ. We shall assume that this region is filled with the same medium that fills R^λ and we shall denote by $\overline{U}^{\lambda}$ and $\overline{W}^{\lambda}$ the functions U and W corresponding to that space for the given medium. In that case, however, the whole space is filled by a single homogeneous medium and the functions W^{λ} and $\overline{W}^{\lambda}$ coincide with the functions V^{λ} and $\overline{V}^{\lambda}$ representing V^{λ} in R^{λ} and $\overline{R}^{\lambda}$. In this way we obtain for $R^0 \equiv R^{\lambda}$ and $R' \equiv \overline{R}^{\lambda}$ from Eq. (9)

$$V^{\lambda} = U^{\lambda} + U^{\lambda}\overline{U}^{\lambda}V^{\lambda}, \quad \overline{V}^{\lambda} = U^{\lambda}\overline{U}^{\lambda} + U^{\lambda}\overline{U}^{\lambda}\overline{V}^{\lambda}.$$

Putting $U^{\lambda}\overline{U}^{\lambda} = \overline{F}^{\lambda}$ we find

$$U^{\lambda} = V^{\lambda} - \overline{F}^{\lambda}V^{\lambda}, \quad F^{\lambda} = \overline{V}^{\lambda} - \overline{F}^{\lambda}\overline{V}^{\lambda}.$$

But from the second equation we obtain directly by integration

$$F^{\lambda} = \overline{V}^{\lambda} - \overline{V}^{\lambda 2} + \overline{V}^{\lambda 3} + \ldots$$

and substituting into the former equation, we obtain finally

$$U^{\lambda} = V^{\lambda} - \overline{V}^{\lambda}V^{\lambda} + \overline{V}^{\lambda 2}V^{\lambda} - \overline{V}^{\lambda 3}V^{\lambda} + \ldots \tag{10}$$

The functions U^{λ}, therefore, have been found.

If we substitute (10) into (7) and (8), and the result into (6), we obtain the functions W in the form of a convergent infinite series, the terms of which are products of the various powers of the known functions V^{λ} and $\overline{V}^{\lambda}$. All the products and powers should of course be understood in the sense of definition (4).

If we consider, for example, the particles of a cosmic ray shower, penetrating a number of plates R^{λ} none of the particles being scattered backwards, then Eqs. (8) and (10) are reduced to:

$$U^{\lambda} = \begin{cases} V^{\lambda} & \text{for } \lambda = \varkappa \\ 0 & \text{for } \lambda \neq \varkappa \end{cases}, \quad W^{\lambda+1}_{m+1} = W^{\lambda}_{m}U^{\lambda+1}.$$

Assuming $\kappa = 0$ and taking into account Eqs. (6) and (7), we obtain the simple relation

$$W^{\lambda} = V^0V^1V^2 \ldots V^{\lambda}.$$

We have assumed that the layers are homogeneous because in that case the functions V can be found. These have been assumed to be known. The above considerations are, however, valid when the layers are not homogeneous.

Bearing in mind first, that integration in Eq. (4) is carried out only over those values of σ for which the development curves are directed from R^{μ} into R^{ν} and, second, that a sharp return of the development curves is unlikely in practice and, third, that the cascade particles are often rapidly absorbed, it can be shown that, in general, the found infinite series converge well.

It should be noted that the products (4) are difficult to calculate since the integrands are complicated functions and the integrals sixtuple. In a number of important cases they can, nevertheless, be markedly simplified, namely:

1. If the medium R^{λ} is a layer of a thickness on the order of the particle mean free path then, instead of U^{λ} we can simply take the expression (1.23) for $m = 1$ and $V^0 = A$. Calculation of the product (4) is then reduced to a double integration (owing to the presence of δ-electrons).

2. If the layers are thick, then it is possible to use expressions of the type (1.59) instead of V^{μ} and V^{ν} in calculating $V^{\mu}V^{\nu}$. Taking it into account that a quadratic function of **r** stands in the exponent of the latter expression it becomes immediately clear that if the surface $S^{\mu\nu}_{j}$ is cylindrical, one integration is reduced to the Poisson integral, and if the surface is a plane two integrations are reduced to that type.

3. As a further approximation in the calculation of the products $V^{\mu}V$ where V^{μ} and V^{ν} are given by (1.59) we can substitute for ϵ, ϵ', ϵ'' and δ', δ''. . . in V^{ν} their values averaged in some sense over **u**. We can, for instance, take the average values over the surface $S^{\mu\nu}_{j}$, calculating by means of V^{μ} the probability of various values of **u** on this surface. The both factors in V^{μ} and V^{ν} will then become exponential functions of quadratic functions of **v**, so that if the surface $S^{\mu\nu}_{j}$ is independent of **u**, as it is practically always the case, then the integration over **u** is reduced to the Poisson integral or to the Gauss function, and the product will be represented by a double integral.

Uniting cases (2) and (3) we obtain that in a multi-layer medium with plane boundaries the products $V^{\mu}V^{\nu}$ are expressed by single integrals which are easy to calculate with the help of computing machines.

The author regards it as a pleasant duty to express his deep gratitude to Academician N. N. Bogoliubov for a valuable discussion of the results, and to Prof. D. D. Ivanenko for his interest in the work.

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CORRELATIONS IN THE DISTRIBUTION OF CASCADE PARTICLES

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A cascade is considered to consist of several types of particles moving in a generally inhomogeneous medium varying with time. The particles collide with the particles of the medium and in this process are absorbed, scattered, and produce new cascade particles. The functions, determining the distribution of the particles of each type in cascades initiated by a single particle of a given type appearing at a given time with known initial position and velocity, are assumed to be known. By means of these functions, the probability of the joint presence of a given number of cascade particles at a given instant in a given cell of the particle type-position-velocity space is found. The detecting probability is calculated for detectors having a sensitivity dependent upon the type and velocity of particles as well as upon their place and time of incidence.

LET $A_j (j = 1, 2, \ldots n)$ represent n various types of particles forming a cascade initiated by a single particle of a given type A_i with a given velocity $\mathbf{u}$, which has appeared at the instant s at a given point $\mathbf{q}$. Let

$$V(P, Q)\,dQ \quad (P = (s, \mathbf{q}, \mathbf{u}, i),\ Q = (t, \mathbf{r}, \mathbf{v}, j),\ dQ = d\mathbf{r}d\mathbf{v}) \tag{1}$$

be the probability that at the time t one particle of this cascade, of the type A_j has the radius vector between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ and velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. This probability was found for the case of a homogeneous[1] and of a multi-layer[2] medium. In the following we shall assume that the functions V are known for any inhomogeneous and time-varying medium without referring to their actual expressions. Using the notation and results of Refs. 1 and 2 we shall solve several generalized problems concerning the correlations in particle distribution.

1. Let L be a natural number. We shall find the probability

$$V(P, Q_1, Q_2, \ldots Q_L)\,dQ_1 dQ_2 \ldots dQ_L \text{ or, in short } V(P, Q_l)\,dQ_l \tag{2}$$

that there are K_ℓ particles of the type $A_{j\ell}$ and with velocity between $\mathbf{v}_\ell$ and $\mathbf{v}_\ell + d\mathbf{v}_\ell$ having at the instants t_ℓ radius vector between $\mathbf{r}_\ell$ and $\mathbf{r}_\ell + d\mathbf{r}_\ell$ respectively.

We shall consider a cascade for which probability (2) is fulfilled, i.e., in which L particles of the type $A_{j\ell}$ with velocities between $\mathbf{v}_\ell$ and $\mathbf{v}_\ell + d\mathbf{v}_\ell$ have radius vectors between $\mathbf{r}_\ell$ and $\mathbf{r}_\ell + d\mathbf{r}_\ell$ at the instants t_ℓ. A certain number of cascade development lines C_ℓ, originating in the point P and passing through or terminating in the points Q_ℓ correspond to those particles. All the lines form a tree-like system R in the space E of the variables t, **r**, **v**, j, beginning with a single branch D_1 emerging from the point P at the time s, which then splits at the time τ_1 into two or more branches D_2, D_3, ... These, in turn, branch out at the moments τ_2, τ_3 ... etc. until all branches terminate in the end points Q_1. We shall denote the branching-out points of the lines D_h by P_k^*. These points will be said to be of multiplicity $M (M = 2, 3, \ldots)$ when M new branches start from a point.

The system R is not continuous since the types and velocities of particles change abruptly in collisions. The projection of R in the t-**r** space is, however, a continuous branching curve. We shall refer to the set of all systems R corresponding to a given position of the points P and Q_ℓ, the projections of which in the t-**r** space are topologically equivalent, as to the graph S. A graph corresponding to the points, P, $Q_1 Q_2, \ldots Q_{10}$, $P_1^* \ldots P_4^*$ is shown in Fig. 1. The points P_1^*, P_3^*, and P_4^* are double, and P_2^* is quadruple. The number N_L of all possible graphs corresponding to a given L is, obviously, finite. For example, $N_1 = 1$, $N_2 = 2$, $N_3 = 3$, etc. All graphs corresponding to the cases L = 1,2,3, are shown in Fig. 2. We shall call a graph S elementary, and denote it by T, if its branches D_h do not pass beyond the points P_ℓ but terminate in them. Every graph S can be decomposed into elementary graphs T, chosing as dividing points those of the points P_ℓ traversed by S. It is obvious that the number of elementary graphs into which every S graph can be divided is also finite—always less than L. The graph represented in Fig. 1 is divided into four elementary graphs by the points Q_1, Q_2, and Q_6.

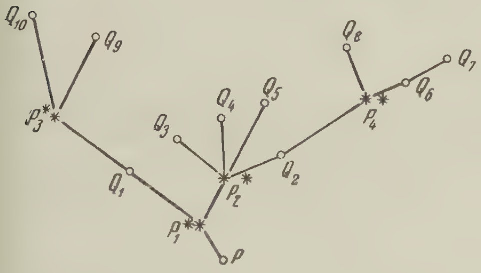

FIG. 1

We shall denote $V_S(P, Q_1)$ $d\mathbf{r}d\mathbf{v}$ the probability (2) with the additional condition that cascade develop according to a given graph S. From elementary theorems on sum and product of probabilities it is evident that the function V is equal to the sum of the functions V_S corrsponding to all graphs S for a given choice of the points P and Q_ℓ, and the function V_S is equal to the product of functions corresponding to all elementatry graphs T into which S may be decomposed.

$$V = \sum_S^L \prod_T^S V_T. \tag{3}$$

Consequently, the solution is reduced to a calculation of the functions V_T corresponding to the elementary graphs.

It is easy to see from the rules for the sum and product of probabilities that V_T represents a sum of multiple integrals, the integrands of which the products of the following factors: a factor

$$V(P', P'')\,dP'' \quad (P' = (\tau', \rho', \boldsymbol{\nu}', k'),\ P'' = (\tau'', \rho'', \boldsymbol{\nu}'', k'')),$$

corresponds to every branch D with the origin in Q′ and end in Q″, and a factor

$$Q^M(P^*, P_m^*)\,d\tau d\mathbf{w}_m \quad (P^* = (\tau, \rho, \boldsymbol{\nu}, k),\ P_m^*(\tau, \rho, \mathbf{w}_m, l_m)),$$

corresponds to every branching point P* of multiplicity M. ℓ_m and $\mathbf{w}_m$ denote the types and velocities of secondary particles (m = 1, 2 ... M). This factor represents the probability that the incident particle will collide with the time $d\tau$ and that among secondary particles there will be M particles of the type ℓ_m with velocity between $\mathbf{w}_m$ and $\mathbf{w}_m + d\mathbf{w}_m$ respectively. Integration and summation is carried out over τ, ρ, ν, k characterizing all the branching points and over the types and velocities of all secondary particles.

Assuming without loss of generality that $t_1 < t_2$, we find, for instance, for L = 2:

$$V(P, Q_1, Q_2) = V(P, Q_1)\,V(Q_1, Q_2) + \sum_{P^*, l_1, l_2} \iint V(P, P^*)\,Q^2(P^*, P_1^*, P_2^*)\,V(P_1^*, Q_1)\,V(P_2^*, Q_2)\,d\mathbf{w}_1 d\mathbf{w}_2.$$

where Σ denotes integration over τ, ρ, ν, and summation over k.

2. In solving the above problem we did not impose any limitation upon either the type of the development graph or the number of collisions in each of its branches. We can modify the problem assuming that the cascade develops according to a given graph and that the number of collisions in some of its branches is pre-determined. It is easy to see that instead of relation (3) we shall have now

$$V = \prod_{T}^{S} V_T. \tag{4}$$

The factor in Eq. (4) corresponding to each branch with a fixed number of collisions will now be, instead of V(P′, P″),

$$\sum_{P''_g l_g} \int W(P', P''_1) \prod_g Q'(P''_g, P'_g) W(P'_g, P''_{g+1}) \, d\mathbf{w}_g$$

$$(P''_g = (\tau_g, \rho_g, \nu_g, k_g),\ P'_g = (\tau_g, \rho_g, \mathbf{w}_g, l_g),\ g = 1, 2, \ldots G - 1),$$

where Q′ and W are given by Eqs. (5) and (6) of Ref. 1 respectively.

If we want to calculate, for example, $V_T(P, Q_1, Q_2, Q_3)$ where T is the last graph of Fig. 2, under the condition that the number of collisions along the first branch is arbitrary and along all other four branches limited to one per branch, we obtain

$$V_T(P, Q_1, Q_2, Q_3) = \sum_{P^*_1 P^*_2 l_{11} l_{12} l_{21} l_{22}} \iiiint V(P, P^*_1)$$
$$\times Q^2(P^*_1, P^*_{11}, P^*_{12}) W(P^*_{11}, P^*_2) Q^2(P^*_2, P^*_{21}, P^*_{22})$$
$$\times W(P^*_{21}, Q_1) W(P^*_{22}, Q_2) W(P^*_{12}, Q_3) \, d\mathbf{w}_{11} d\mathbf{w}_{12} d\mathbf{w}_{21} d\mathbf{w}_{22}.$$

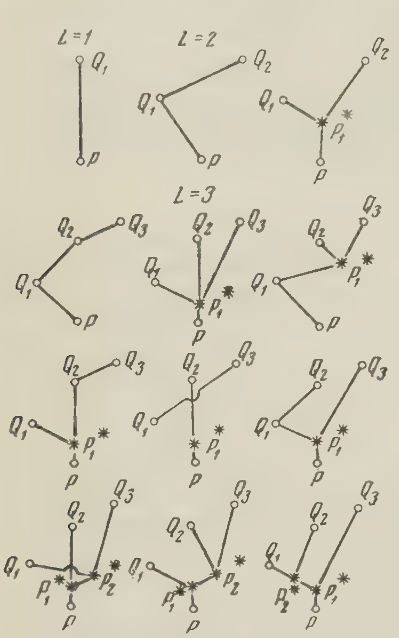

FIG. 2

3. In deducing the expressions (3) and (4) we assumed that the regions in which the particles P_ℓ are found are infinitesimally small. We shall generalize the problem for the case when the regions are finite. Let $t_\ell (\ell = 1, 2, \ldots, L)$ be L moments of time, let R_ℓ be L regions in the $\mathbf{r} - \mathbf{v} - \mathrm{j}$ space, and let N_ℓ be L positive integers. We shall denote by

$$W(P, t_l, R_l, N_l) \tag{5}$$

the required probability of finding N_ℓ particles at the moments t_ℓ and in the regions R_ℓ respectively.

Let M_ℓ be L positive integers and let $\mathbf{r}_{\ell m}, \mathbf{v}_{\ell m}, \mathrm{j}_{\ell m} (m = 1, 2, \ldots, M_\ell)$ represent the coordinates of M_ℓ points in the region R_ℓ. Let

$$V(P, Q_{lm}) dQ_{lm} \quad (l = 1, 2, \ldots L, m = 1, 2, \ldots M_l)$$

be the probability of the type (2) for $L = \Sigma M_\ell$ and $t_{\ell m} = t_\ell$.

We shall denote by

$$V(P, t_l, R_l, M_l) = \frac{1}{\prod_l M_l!} \sum_{j_{lm}} \int V(P, Q_{lm}) dQ_{lm} \tag{6}$$

the probability that there will be at least M_ℓ particles in the regions R_ℓ at the moments t_ℓ respectively. [In Eq. (6) and in the following $dQ_{\ell m} = d\mathbf{r}_{\ell m} d\mathbf{v}_{\ell m}$, and the integration is carried out over the part of the $\mathbf{r}_{\ell m}\mathbf{v}_{\ell m}$ plane contained in R_ℓ].

In order to find the relation between (5) and (6) we shall solve first an auxiliary problem. Let there be given I cells $C_i (i = 1, 2, \ldots, I)$ such that there can be at most one particle in each of them. Let the appearance of particles in the different cells be uncorrelated. Let P and Q be positive integers not greater than I. Let $i_p (p = 1, 2, \ldots, P)$ and $j_q (q = 1, 2, \ldots, Q)$ be two sequences of different natural numbers ≤ I. We shall denote by $V(i_p)$ the probability that the cells C_i are occupied regardless of whether the other cells are occupied or not, and by $W(j_q)$ the probability that the cells C_{jp} are occupied while all other cells are free. Both these functions are symmetric with respect to all arguments. It is evident that

$$V_P = \frac{1}{P!} \sum_{i_p} V(i_p) \quad \text{and} \quad W_Q = \sum_{j_q} W(j_q) - \tag{7}$$

represent, respectively, the probability that not less than P or exactly Q cells are occupied. We shall show that the following relation holds:

$$V(i_p) = \frac{1}{(Q-P)!} \sum_{j_q}^{q>P} W(j_q) \quad (j_q = i_q \text{ for } q \leqslant P), \tag{8}$$

where the summation is carried out over all j_q for $q < P$. Eq. (8) expresses the elementary fact that the probability that the cells C_{ip} are occupied regardless whether other cells are occupied equals the sum of the occupation probabilities for all combinations of the cells C_i containing the cells C_{ip}. The factor $1/(Q-P)$ is due to indeterminate grouping of the numbers j_q. Summing over all i_p and taking into account Eq. (7), the relation (8) leads to

$$V_P = \sum_{Q=P}^{\infty} \frac{Q!}{Q!\,(Q-P)!} W_Q \quad (P = 0, 1, \ldots).$$

This equation represents an infinite system of equations for W_Q which can be solved easily, yielding

$$W_Q = \sum_{P=Q}^{\infty} \frac{(-1)^{P-Q}\, P!}{Q!\,(P-Q)!} V_P \quad (Q = 0, 1, \ldots). \tag{9}$$

The values (5) and (6) are analogous to those of Eq. (9). In consequence they are connected by a relation similar to (9)

$$W(P, t_l, R_l, N_l) = \sum_{M_l = N_l}^{\infty} \prod_{l=1}^{L} \frac{(-1)^{M_l - N_l} M_l!}{N_l!\,(M_l - N_l)!} V(P, t_l, R_l, M_l). \tag{10}$$

Equation (10) is more complicated than Eq. (9) since we are dealing now with L regions R_ℓ instead of one.

In calculating the probability (5) we did not require the N_ℓ particles found in R_ℓ to be of a certain type. The solution, however, is more general — it is sufficient to divide the region R into n regions $R_{\ell j}$ in order to find the probability that given numbers of particles $N_{\ell j}$ ($j = 1, 2, \ldots n$) of each type A_j are present in R_ℓ.

4. We can generalize the obtained solution further if we replace the regions R_ℓ by detecting devices, acting at the moments t_ℓ respectively, each of which detects the traversing particles with a certain probability $U^\ell(Q)$ depending on the device R_ℓ, on the time t, on the type A_j of the particle, on its velocity v, and on its position r. The probability (5) is in this case expressed again by Eq. (9), V being given now, however, not by Eq. (6) but by a more general formula

$$V(P, t_l, R_l, M_l) = \frac{1}{\prod_l M_l!} \sum_{j_{lm}} \int V(P, Q_{lm}) \prod_{lm}^{LM_l} U^l(Q_{lm})\, dQ_{lm}. \tag{11}$$

5. The various types of counters used for the detection of cosmic rays do not fall exactly into the category of devices of Sec. 4. The actual detectors function continuously or during a certain time interval, while above it was assumed that the particles were detected at certain instants. We shall consider therefore another type of devices R_ℓ ($\ell = 1, 2, \ldots L$) which can time the arrival of particles. Each device is characterized by a certain working volume R_ℓ, which, for the sake of symmetry, we shall consider as a volume in the space E, and by a certain function $U^\ell(Q)$ defined on a surface S^ℓ of the region R_ℓ or, which is equivalent, on surfaces S_j^ℓ in E, since $U_j^\ell d\sigma$ yields the detection probability of a particle of the type A_j entering R_ℓ through an element $d\sigma$ of the surface S_j^ℓ corresponding to the values σ and $\sigma + d\sigma$ of the six-dimensional variable σ. Evidently, this assumption represents an approximation since in view of the finite resolving power of the detectors, the detection probability U^ℓ can also depend on the sequence of arrival of various particles at R_ℓ. In this approximation the probability V is again given by Eq. (11), it is necessary, however, to write according to Eq. (4) of Ref. 2, instead of $V(P, Q_{\ell m})$, the following function:

$$V^*(P, Q_{lm}) = V(P, Q_{lm}) \begin{vmatrix} 1 & \mathbf{v}_{lm} & \mathbf{F}_j(t_l, \mathbf{r}_{lm}, \mathbf{v}_{lm}) \\ \dfrac{\partial t_l}{\partial \sigma} & \dfrac{\partial \mathbf{r}_{lm}}{\partial \sigma} & \dfrac{\partial \mathbf{v}_{lm}}{\partial \sigma} \end{vmatrix},$$

and the integration and summation should be carried out over the surfaces S^{ℓ}.

V being known, we can find the required probability (5) by means of relation (9) which remains valid.

All the probabilities found above may be useful, for example, in investigations of the cosmic radiation by means of coincidence counters or cloud chambers.[3]

It should be noted that the probabilities found are analogous to the probabilities of certain configurations of molecules in a gaseous medium. In the case studied above, however, the problem is greatly simplified since all the probabilities can be expressed by means of the distribution function (1), while there are no similar expressions for the correlation function in gasses.[4]

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POLARIZATION OF NUCLEONS ELASTICALLY SCATTERED AGAINST TARGET PARTICLES OF SPIN 1

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The average values of the spin operators for a system of particles having spin 1 and $^1/_2$ are calculated. The transition matrix M is given explicitely. Consideriation is given to the case of small energies, when one can restrict oneself to S- and P-waves. Expressions are obtained for the cross-section, polarization, and correlation function. Relationships are established between the parameters of the transition matrix and the experimentally observed values. A group of experiments is suggested which could enable one to determine, through triple-scattering, the amplitude of the scattered wave and to carry out a phase-shift analysis.

THE theory of reactions involving polarized nucleons has been recently developed in a series of articles.[1] The polarization arising in nucleon-nucleon collision is due to spin-orbit interaction, and its measurement provides additional information about the coefficients of the amplitude for nucleon-nucleon scattering. A group of experiments is indicated which would allow one to determine the nucleon-nucleon scattering amplitude and to carry out a phase-shift analysis.

The present article is concerned with the elastic scattering of nucleons against a target made up of spin 1 particles.

The state of the system is described as usual through the Neuman density matrix ρ in the combined spin space of the system of two particles, or through the density matrix for two independent beams of free particles with spin $\boldsymbol{\sigma}'$ and $\mathbf{s}'$. Since the density matrix is a six-rowed Hermitian matrix, it is defined through a linear combination of 36 real quantities for which one may take the average values of a complete set of independent Hermitian operators in spin space (their number equals the square of the dimensionality of the spin space). The complete set of operators satisfies the equations

$$\mathrm{Sp}\, S^{\mu} S^{\nu} = (2s_a + 1)(2s_b + 1)\,\delta_{\mu\nu}. \tag{1}$$

The average value of any operator is defined as

$$< S >_{\mathrm{inc}} = \mathrm{Sp}\,(\rho_{\mathrm{inc}} S) \tag{2}$$

with the condition that

$$\mathrm{Sp}\,\rho_{\mathrm{inc}} = 1.$$

An arbitrary matrix may be linearly expressed in terms of the operators S^{μ}. Expressing the density matrix in terms of the average values of the spin operators, we obtain

$$\rho_{\mathrm{inc}} = [(2s_a + 1)(2s_b + 1)]^{-1} \sum_{\mu} < S^{\mu} > S^{\mu}. \tag{3}$$

In this way, the average value of the matrix S^{μ} determines ρ, and, at the same time, the spin state of the system.

If we know the transition matrix M which transforms each pure state of a mixture described by some matrix ρ_{inc}, into some corresponding state of the mixture described by ρ_{scat}, we then have for the scattering of a beam

$$\rho_{\mathrm{scat}} = M \rho_{\mathrm{inc}} M^{*}. \tag{4}$$

The average value of any operator S_1 in the spin space of particles is given, after collision, by the expression

$$< S_1 >_{\mathrm{scat}} = \mathrm{Sp}\,(\rho_{\mathrm{scat}} S_1) \,/\, \mathrm{Sp}\,\rho_{\mathrm{scat}}. \tag{5}$$

The following expressions are then found for the differential reaction cross section and the average value of the operator S_1:

$$Q = [(2s_a + 1)(2s_b + 1)]^{-1} \sum_{\mu} < S^{\mu} >_{\mathrm{inc}} \mathrm{Sp}\,(M S^{\mu} M^{*}), \tag{6}$$

$$< S_1 >_{\mathrm{scat}} = \sum_{\mu} < S^{\mu} >_{\mathrm{inc}} \mathrm{Sp}\,(M S^{\mu} M^{*} S_1) \Big/ \sum_{\nu} < S^{\nu} >_{\mathrm{inc}} \mathrm{Sp}\,(M S^{\nu} M^{*}). \tag{7}$$

THE AVERAGE VALUES OF THE SPIN OPERATORS

Applying the well-known Clebsch-Gordan formulas, the spin-quartet wave function may be written in the form

$$\begin{gathered} \chi({}^3\!/_2, {}^3\!/_2) = \chi({}^1\!/_2, {}^1\!/_2)\,\chi(1, 1), \quad \chi({}^3\!/_2, {}^1\!/_2) = \sqrt{{}^2\!/_3}\,\chi({}^1\!/_2, {}^1\!/_2)\,\chi(1,0) + \sqrt{{}^1\!/_3}\,\chi({}^1\!/_2, -{}^1\!/_2)\,\chi(1, 1), \\ \chi({}^3\!/_2, -{}^1\!/_2) = \sqrt{{}^1\!/_3}\,\chi({}^1\!/_2, {}^1\!/_2)\,\chi(1, -1) + \sqrt{{}^2\!/_3}\,\chi({}^1\!/_2, -{}^1\!/_2)\,\chi(1, 0), \quad \chi({}^3\!/_2, -{}^3\!/_2) = \chi({}^1\!/_2, -{}^1\!/_2)\,\chi(1, -1), \end{gathered} \tag{8}$$

and the doublet function becomes

$$\begin{gathered} \chi({}^1\!/_2, {}^1\!/_2) = \sqrt{{}^1\!/_3}\,\chi({}^1\!/_2, {}^1\!/_2)\,\chi(1, 0) - \sqrt{{}^2\!/_3}\,\chi({}^1\!/_2, -{}^1\!/_2)\,\chi(1, 1), \\ \chi({}^1\!/_2, -{}^1\!/_2) = \sqrt{{}^2\!/_3}\,\chi({}^1\!/_2, {}^1\!/_2)\,\chi(1, -1) - \sqrt{{}^1\!/_3}\,\chi({}^1\!/_2, -{}^1\!/_2)\,\chi(1,0). \end{gathered} \tag{9}$$

Operating on these functions with the operators σ_x, σ_y, σ_z, S_x S_y and S_z, and applying the orthonormality of the functions $\chi\,(s, m_s)$, one obtains

$$\sigma_x = \left\{ \begin{matrix} 0 & \sqrt{1/3} & 0 & 0 & -\sqrt{2/3} & 0 \\ \sqrt{1/3} & 0 & 2/3 & 0 & 0 & -\sqrt{2}/3 \\ 0 & 2/3 & 0 & \sqrt{1/3} & \sqrt{2}/3 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & 0 & \sqrt{2/3} \\ -\sqrt{2/3} & 0 & \sqrt{2}/3 & 0 & 0 & -1/3 \\ 0 & -\sqrt{2}/3 & 0 & \sqrt{2/3} & -1/3 & 0 \end{matrix} \right\}, \quad \sigma_y = -\left\{ \begin{matrix} 0 & -\sqrt{1/3} & 0 & 0 & \sqrt{2/3} & 0 \\ \sqrt{1/3} & 0 & -2/3 & 0 & 0 & \sqrt{2}/3 \\ 0 & 2/3 & 0 & -\sqrt{1/3} & \sqrt{2}/3 & 0 \\ 0 & 0 & \sqrt{1/3} & 0 & 0 & \sqrt{2/3} \\ -\sqrt{2/3} & 0 & -\sqrt{2}/3 & 0 & 0 & 1/3 \\ 0 & -\sqrt{2}/3 & 0 & -\sqrt{2/3} & -1/3 & 0 \end{matrix} \right\},$$

$$S_x = \frac{1}{\sqrt{2}} \left\{ \begin{matrix} 0 & \sqrt{2/3} & 0 & 0 & \sqrt{1/3} & 0 \\ \sqrt{2/3} & 0 & 2\sqrt{2}/3 & 0 & 0 & 1/3 \\ 0 & 2\sqrt{2}/3 & 0 & \sqrt{2/3} & -1/3 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & 0 & -\sqrt{1/3} \\ \sqrt{1/3} & 0 & -1/3 & 0 & 0 & 2\sqrt{2}/3 \\ 0 & 1/3 & 0 & -\sqrt{1/3} & 2\sqrt{2}/3 & 0 \end{matrix} \right\}, \quad S_z = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & -\sqrt{2}/3 & 0 \\ 0 & 0 & -1/3 & 0 & 0 & -\sqrt{2}/3 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\sqrt{2}/3 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & -\sqrt{2}/3 & 0 & 0 & -2/3 \end{matrix} \right\}.$$

By means of a suitable transformation, one may go from this representation to a 2-particle representation which may be more easily obtained directly as a direct product of the matrices σ' and S' in the spin-spaces corresponding to a nucleon and a deuteron.

We shall denote unit vectors in the direction of motion of the incident and scattered beams in the center-of-mass system respectively by $\mathbf{k}_i$ and $\mathbf{k}_f$, and for convenience in future expansions, we shall introduce as usual the orthogonal vectors

$$\mathbf{N} = [\mathbf{k}_i \times \mathbf{k}_f] / |[\mathbf{k}_i \times \mathbf{k}_f]|, \quad \mathbf{P} = (\mathbf{k}_i + \mathbf{k}_f) / |\mathbf{k}_i + \mathbf{k}_f|, \quad \mathbf{K} = (\mathbf{k}_i - \mathbf{k}_f) / |\mathbf{k}_i - \mathbf{k}_f|. \tag{10}$$

Transforming into this system by means of Euler's formulas,[3] we obtain formulas for σ_N, σ_p, σ_K, S_N, S_p, S_K expressed in terms of the angles ϑ and φ; we shall not list them here, as they are rather cumbersome.

THE TRANSITION MATRIX

The most general form of the matrix M, which determines the amplitude of a scattered wave of given spin and momentum as a function of the spin and momentum of the incident wave, may be obtained by imposing upon it the conditions of invariance with respect to space rotation and time reversal. This matrix must be a scalar which is obtained from a combination of 36 linearly independent matrices in spin space and functions of the momenta **K, N, P.** The effect of time reversal leads to the transformation

$$\sigma' = -\sigma; \quad \mathbf{k}_i' = -\mathbf{k}_f; \quad \mathbf{k}_f' = -\mathbf{k}_i; \quad \mathbf{K}' = \mathbf{K}; \quad \mathbf{N}' = -\mathbf{N}; \quad \mathbf{P}' = -\mathbf{P}.$$

The expressions $[\sigma \times \mathbf{S}]\mathbf{N}$ and $(\sigma\mathbf{K})(\mathbf{SP}) + (\sigma\mathbf{P})(\mathbf{SK})$ change sign with time reversal and therefore must be excluded from consideration. In writing out the matrix M, one may limit oneself to terms of second order in S, as higher order terms in the spin operator S may be given as functions of these. Thus, for example

$$S_i S_j S_k = \frac{1}{2}[\delta_{ij} S_k + \delta_{jk} S_i] + \frac{i}{2}[S_{i\times j,k} + S_{j\times k,i} - S_{k\times i,j}] + \frac{i}{3}\varepsilon_{ijk},$$

$$S_{ij} S_{kl} = \frac{1}{3}\left[\delta_{ij} S_{kl} + \delta_{kl} S_{ij} + \frac{1}{2}\delta_{ik}\delta_{jl} + \frac{1}{2}\delta_{jk}\delta_{il} - \frac{1}{3}\delta_{ij}\delta_{kl}\right]$$

$$-\frac{1}{4}\left[\delta_{ik}\left(S_{jl} - \frac{i}{2}S_{j\times l}\right) + \delta_{il}\left(S_{jk} - \frac{i}{2}S_{j\times k}\right) + \delta_{jk}\left(S_{il} - \frac{i}{2}S_{i\times l}\right) + \delta_{jl}\left(S_{ik} - \frac{i}{2}S_{i\times k}\right)\right],$$

where

$$S_{j\times k} = \varepsilon_{ijk} S_i, \quad S_{j\times k,l} = \varepsilon_{ijk} S_{il}.$$

In this fashion one obtains the most general form of the scattering matrix for particles of spin 1/2 against particles of spin 1:

$$
\begin{aligned}
M = A + BS_N + C\,[(S_P S_P - {}^2/_3\,\delta_{ij}) + (S_K S_K - {}^2/_3\,\delta_{ij})]\\
+ \{A_1 + B_1 S_N + C_1\,[(S_P S_P - {}^2/_3\,\delta_{ij}) + (S_K S_K - {}^2/_3\,\delta_{ij})]\}\,\sigma_N + D\,(\sigma_P S_P + \sigma_K S_K)\\
+ E\,(\sigma_P S_P - \sigma_K S_K) + {}^1/_2\,F\,(\sigma_P S_N S_P + \sigma_K S_N S_K) + {}^1/_2 G\,(\sigma_P S_P S_N + \sigma_K S_K S_N)\\
+ {}^1/_2\,H\,(\sigma_P S_N S_P - \sigma_K S_N S_K) + {}^1/_2\,K\,(\sigma_P S_P S_N - \sigma_K S_K S_N),
\end{aligned}
\tag{11}
$$

where the coefficients A, B, C etc. appear as functions of k_i^2 and $\mathbf{k}_i\mathbf{k}_f$, i.e., as functions of the energy and the cosine of the scattering angle in the center-of-mass system.

In order to obtain the matrix in explicit form, we apply the method of Blatt and Bienderharn.[4] The amplitude of the scattered wave is given as the sum of two terms:

$$f(\vartheta) = f_1(\vartheta) + f_2(\vartheta).$$

The first of these

$$f_1(\vartheta) = (-\lambda\eta / 2\zeta^2)\exp[-i\eta\ln\zeta^2] \tag{12}$$

depends on Coulomb scattering, while the second

$$
\begin{aligned}
f(s') = i\lambda\Phi_{s'} \sum_{m_{s'}} \chi(s', m_{s'}) \sum_{Jll'} i^{l-l'}\pi^{1/2}(2l+1)^{1/2}(l, s, 0, m_s \,|\, l, s, J, m_s)\\
\times (l', s', m_s - m_{s'}, m_{s'} \,|\, l, s', J, m_s)\,(\delta_{s's}\delta_{l'l} - S^J_{s'l';\,sl})\,Y_{l', m_s - m_{s'}}(\vartheta, \varphi) = \sum_{m_{s'}} \chi(s', m_{s'}) \sum_{m_s} M_{m_{s'} m_s} a_{m_s}
\end{aligned}
\tag{13}
$$

gives the difference between the amplitude of a wave scattered against a charged sphere and $f_1(\vartheta)$. In the above expression we have used certain quantities defined as follows: $\eta = e^2/hv$; $\zeta = \sin(\vartheta/2)$; $\Phi_{s'}$ is the product of the wave functions of the final nucleus and the scattered particle; the quantities a_{m_s} obey the condition

$$a_{m_{s'}}a_{m_s} = \delta(m_s - m_{s'})/(2s+1). \quad S^J_{s',\,l';\,s,\,l} = \delta_{s',\,s}\delta_{l',\,l}\exp[2i(\xi_l - \eta\ln 2kr)]$$

are the elements of the scattering matrix, r is the radius of the screened Coulomb field in a given channel and

$$\xi_l = \psi_l + \Phi_l + \sigma_0,$$

where $\psi_\ell = \sigma_\ell - \sigma_0$ is the Coulomb phase shift:

$$\exp(2i\psi_l) = (l + i\eta)\ldots(1 + i\eta)/(l - i\eta)\ldots(1 - i\eta),$$

σ_0 is the S-wave phase shift; $\Phi_\ell = \xi_\ell - \sigma_\ell$ describes a further phase shift occurring during scattering against a charged sphere of radius R as compared with the shift which occurs in point charge scattering.

In order to carry out a phase-shift analysis, it is necessary to determine the dependence of M upon the scattering angle and the phase shift. The dependence of the elements of the matrix M upon φ is evident from Eq. (13), and comparing them with those matrices of which it consists, we obtain

$$
M = \lambda \left\{ \begin{array}{cccccc}
a & ce^{-i\varphi} & ge^{-2i\varphi} & he^{-3i\varphi} & 0 & 0\\
be^{i\varphi} & d & fe^{-i\varphi} & ge^{-2i\varphi} & 0 & 0\\
ge^{2i\varphi} & -fe^{i\varphi} & d & -be^{-i\varphi} & 0 & 0\\
-he^{3i\varphi} & ge^{2i\varphi} & -ce^{i\varphi} & a & 0 & 0\\
0 & 0 & & 0 & \mu & -\nu e^{-i\varphi}\\
0 & 0 & 0 & 0 & \nu e^{i\varphi} & \mu
\end{array} \right\}.
\tag{14}
$$

the matrix elements a, b, c represent series of Legendre polynomials $P_\ell^m(\cos\vartheta)$, and the coefficients in these series depend on the phase shifts.

The elements of this matrix obey the additional condition

$$\frac{1}{\cos 2\vartheta}\left(a - d - \frac{2}{\sqrt{3}}\,g\right) = \frac{2}{\sqrt{3}}\frac{1}{\sin 2\vartheta}(b + c), \tag{15}$$

which appears as a result ot the invariance of M with respect to time reversal. As mentioned above, the matrix $\sigma K\cdot SP + \sigma P\cdot SK$ is one of the matrices belonging to the complete set in terms of which M may be expanded; thus it can only enter in the expansion when Sp M $(\sigma K\cdot SP + \sigma P\cdot SK)$ is identically zero.

In the particular can when the energy is sufficiently small to permit the use of S- and P-waves only, the scattering amplitude may be written in the form:

$$\psi(s, s') = \frac{e^{ikr}}{2ikr}\sum_{m_s m_{s'}}\sum_{L,J}[4\pi(2L+1)]^{1/2}(L, s, 0, m_s\,|\,L, s, J, m_s)$$
$$\times (L, s, m_s - m_{s'}, m_{s'}\,|\,L, s', J, m_s)\,[1-\exp(2i\xi_l)]\,Y_{L,\,m_s-m_{s'}}(\vartheta, \varphi). \tag{16}$$

For doublet transitions we obtain

$$M_{1/2,1/2} = -\lambda\{\gamma_{0,1/2}^{1/2} + \cos\vartheta e^{-2i\psi}(\gamma_{1,1/2}^{1/2} + 2\gamma_{1,1/2}^{3/2})\},\quad M_{-1/2,1/2} = \lambda\sin\vartheta\exp[i(\varphi-2\psi)](\gamma_{1,1/2}^{1/2} - \gamma_{1,1/2}^{3/2}),$$
$$M_{1/2,-1/2} = \lambda\sin\vartheta\exp[-i(\varphi+2\psi)](-\gamma_{1,1/2}^{1/2} + \gamma_{1,1/2}^{3/2}),\ M_{-1/2,-1/2} = \lambda\{\gamma_{0,1/2}^{1/2} + \cos\vartheta\exp(-2i\psi)(\gamma_{1,1/2}^{1/2} + 2\gamma_{1,1/2}^{3/2}), \tag{17}$$

where $\gamma_{\ell,s}^{J} = \exp(-\Phi_{\ell,s}^{J})\sin\Phi_{\ell,s}^{J}$,

and for quartets

$$M_{3/2,3/2} = -\lambda\left\{\gamma_{0,3/2}^{3/2} + \cos\vartheta\exp(-2i\psi)\left(\frac{9}{5}\gamma_{1,3/2}^{3/2} + \frac{6}{5}\gamma_{1,3/2}^{5/2}\right)\right\},\quad M_{1/2,3/2} = \frac{3\sqrt{3}}{5}\lambda\exp[i(\varphi-2\psi)]\sin\vartheta(\gamma_{1,3/2}^{3/2} - \gamma_{1,3/2}^{5/2}),$$
$$M_{-1/2,3/2} = 0,\qquad M_{-3/2,3/2} = 0;\quad M_{3/2,1/2} = \lambda\sin\vartheta\exp[-i(\varphi+2\psi)]\left(-\frac{\sqrt{3}}{2}\gamma_{1,3/2}^{1/2} + \frac{\sqrt{3}}{5}\gamma_{1,3/2}^{3/2} + \frac{3\sqrt{3}}{10}\gamma_{1,3/2}^{5/2}\right),$$
$$M_{1/2,1/2} = -\lambda\left\{\gamma_{0,3/2}^{3/2} + \cos\vartheta\exp(-2i\psi)\left(\gamma_{1,3/2}^{1/2} + \frac{1}{5}\gamma_{1,3/2}^{3/2} + \frac{9}{5}\gamma_{1,3/2}^{5/2}\right)\right\},$$
$$M_{-1/2,1/2} = \lambda\sin\vartheta\exp[i(\varphi-2\psi)]\left(\frac{1}{2}\gamma_{1,3/2}^{1/2} + \frac{2}{5}\gamma_{1,3/2}^{3/2} - \frac{9}{10}\gamma_{1,3/2}^{5/2}\right),\quad M_{-3/2,1/2} = 0;\ M_{3/2,-1/2} = 0, \tag{18}$$
$$M_{1/2,-1/2} = -\lambda\sin\vartheta\exp[-i(\varphi+2\psi)]\left(\frac{1}{2}\gamma_{1,3/2}^{1/2} + \frac{2}{5}\gamma_{1,3/2}^{3/2} - \frac{9}{10}\gamma_{1,3/2}^{5/2}\right),\quad M_{-1/2,-1/2} = -\lambda\{\gamma_{0,3/2}^{3/2} + \cos\vartheta e^{-2i\psi}$$
$$\times\left(\gamma_{1,3/2}^{1/2} + \frac{1}{5}\gamma_{1,3/2}^{3/2} + \frac{9}{5}\gamma_{1,3/2}^{5/2}\right),\quad M_{-3/2,-1/2} = \lambda\exp[i(\varphi-2\psi)]\sin\vartheta\left(\frac{\sqrt{3}}{2}\gamma_{1,3/2}^{1/2} - \frac{\sqrt{3}}{5}\gamma_{1,3/2}^{3/2} - \frac{3\sqrt{3}}{10}\gamma_{1,3/2}^{5/2}\right);$$
$$M_{3/2,-3/2} = 0,\ M_{1/2,-3/2} = 0,\quad M_{-1/2,-3/2} = -\frac{3\sqrt{3}}{5}\lambda\sin\vartheta\exp[-i(\varphi+2\psi)](\gamma_{1,3/2}^{3/2} - \gamma_{1,3/2}^{5/2}),$$
$$M_{-3/2,-3/2} = \lambda\left\{\gamma_{0,3/2}^{3/2} + \cos\vartheta e^{-2i\psi}\left(\frac{9}{5}\gamma_{1,3/2}^{3/2} + \frac{6}{5}\gamma_{1,3/2}^{5/2}\right)\right\}.$$

DETERMINATION OF THE PARAMETERS OF THE TRANSITION MATRIX FROM EXPERIMENTAL DATA

Expressions (6) and (7) apply to particles of arbitrary spin and beams of arbitrary polarization; we shall apply them to the concrete case of scattering of particles having spin 1/2 against an unpolarized target of particles with spin 1.

In this case, the only non-zero matrices before collision are 1×1 and $\sigma\times 1$, and these formulas become

$$Q^{P} = \frac{1}{6}\{\mathrm{Sp}(MM^*) + \mathrm{Sp}(M\sigma M^*)\mathbf{P}_{\mathrm{inc}}\},\quad \mathbf{P}^{P} = \langle\sigma\times 1\rangle_{\mathrm{scat}} = \frac{\mathrm{Sp}(M^*\sigma M) + \mathrm{Sp}(\boldsymbol{\sigma}M\sigma\mathbf{P}_{\mathrm{inc}}M^*)}{\mathrm{Sp}(MM^*) + \mathrm{Sp}(M\sigma M^*)\mathbf{P}_{\mathrm{inc}}}. \tag{19}$$

We write here σ for $\sigma\times 1$, and denote $\langle\sigma\times 1\rangle_{\mathrm{inc}}$ by $\mathbf{P}_{\mathrm{inc}}$. When scattering an unpolarized beam of nucleons, the expressions for the differential cross-section and the polarization simplify to

$$Q^{N} = \frac{1}{6}\mathrm{Sp}\,MM^*,\quad Q^{N}P^{N} = \frac{1}{6}\mathrm{Sp}\,MM^*\sigma_N \tag{20}$$

Substituting Eq. (14) for the transition matrix M, and expressing σ_N in terms of the Eulerian angles, we obtain explicit forms for the cross-section and the polarization:

$$Q^{N} = \frac{1}{3}\lambda^2\{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |f|^2 + 2|g|^2 + |h|^2 + |\mu|^2 + |\nu|^2\},$$
$$Q^{N}P^{N} = \frac{2}{3}\lambda^2\,\mathrm{Im}\left\{\sqrt{\frac{1}{3}}(ba^* + dc^* + fg^* + gh^*) + \frac{2}{3}(bg^* + df^*) + \frac{1}{3}\mu\nu^*\right\}. \tag{21}$$

In the particular case of small energies, limiting ourselves to S- and P-waves, we obtain for the scattering of a proton against a deuteron:

$$Q^{N}=\lambda^{2}\Big\{\Big(\frac{\eta}{2}\frac{1}{\zeta^{2}}\Big)^{2}+\frac{2}{3}\Big(\frac{\eta}{2}\frac{1}{\zeta^{2}}\Big)[u(\zeta,\Phi^{1/2}_{0,1/2})+2u(\zeta,\Phi^{3/2}_{0,3/2})]+\frac{2}{3}\frac{\eta}{2\zeta^{2}}\cos\vartheta\,[u(\zeta,\Phi^{1/2}_{1,1/2})+2u(\zeta,\Phi^{3/2}_{1,1/2})+u(\zeta,\Phi^{1/2}_{1,3/2})+2u(\zeta,\Phi^{3/2}_{1,3/2})$$

$$+3u(\zeta,\Phi^{5/2}_{1,3/2})]+\frac{1}{3}\sin^{2}\Phi^{1/2}_{0,3/2}+\frac{2}{3}\sin^{2}\Phi^{3/2}_{0,3/2}+\frac{2}{3}\cos\vartheta\,\{\sin\Phi^{1/2}\,[u(\Phi^{1/2}_{0,1/2},\Phi^{1/2}_{1,1/2})$$

$$+2u(\Phi^{1/2}_{0,1/2},\Phi^{3/2}_{1,1/2})]+\sin\Phi^{3/2}_{0,3/2}[u(\Phi^{3/2}_{0,3/2},\Phi^{1/2}_{1,3/2})+2u(\Phi^{3/2}_{0,3/2},\Phi^{3/2}_{1,3/2})+3u(\Phi^{3/2}_{0,3/2},\Phi^{5/2}_{1,3/2})]+\cos^{2}\vartheta\,[(\sin^{2}\Phi^{1/2}_{1,1/2}+\sin^{2}\Phi^{1/2}_{1,3/2})$$

$$+2(\sin^{2}\Phi^{3/2}_{1,1/2}+\sin^{2}\Phi^{3/2}_{1,3/2})+3\sin^{2}\Phi^{5/2}_{1,3/2}-2W]+\sin^{2}\vartheta\cdot W\},$$

where $u(\alpha^{j}_{l,s};\beta^{j'}_{l',s'})=\sin\beta^{j'}_{l',s'}\cos(\alpha^{j}_{l,s}-2l'\psi-\beta^{j'}_{l',s'})$ and

$$W=\frac{1}{3}\sin^{2}(\Phi^{3/2}_{1,1/2}-\Phi^{1/2}_{1,1/2})+\frac{1}{30}\sin^{2}(\Phi^{1/2}_{1,3/2}-\Phi^{3/2}_{1,3/2})+\frac{3}{10}\sin^{2}(\Phi^{1/2}_{1,3/2}-\Phi^{5/2}_{1,3/2})+\frac{21}{50}\sin^{2}(\Phi^{3/2}_{1,3/2}-\Phi^{5/2}_{1,3/2}); \qquad (22)$$

$$Q^{N}P^{N}=\frac{\lambda^{2}}{3}\frac{\eta}{2\zeta^{2}}\sin\vartheta\,\{5z(\Phi^{5/2}_{1,3/2},\Phi^{1/2}_{1,3/2},\zeta)+4z(\Phi^{5/2}_{1,3/2},\Phi^{3/2}_{1,3/2},\zeta)+2z(\Phi^{3/2}_{1,1/2},\Phi^{1/2}_{1,1/2},\zeta)\}+\frac{\lambda^{2}}{3}\sin\vartheta\,\{[5z(\Phi^{1/2}_{1,3/2},\Phi^{1/2}_{1,3/2},\Phi^{3/2}_{0,3/2})$$

$$+4z(\Phi^{5/2}_{1,3/2},\Phi^{3/2}_{1,3/2},\Phi^{3/2}_{0,3/2})]\sin\Phi^{3/2}_{0,3/2}+2z(\Phi^{3/2}_{1,1/2},\Phi^{1/2}_{1,1/2},\Phi^{1/2}_{0,1/2})\sin\Phi^{1/2}_{0,1/2}\}+\frac{\lambda^{2}}{5}\sin\vartheta\cos\vartheta\sin(\Phi^{1/2}_{1,3/2},-\Phi^{3/2}_{1,3/2})\sin(\Phi^{1/2}_{1,3/2}-\Phi^{5/2}_{1,3/2}).$$

$$\times\sin(\Phi^{3/2}_{1,1/2}-\Phi^{5/2}_{1,3/2})-\lambda^{2}\sin\vartheta\cos\vartheta\,[5Y(\Phi^{1/2}_{1,3/2},\Phi^{5/2}_{1,3/2})+4Y(\Phi^{3/2}_{1,3/2},\Phi^{5/2}_{1,3/2})+2Y(\Phi^{1/2}_{1,1/2}\Phi^{3/2}_{1,1/2})],$$

where $Y(\alpha,\beta)=\sin(\alpha-\beta)\sin\alpha\cdot\sin\alpha$, and $z(\alpha,\beta,\gamma)=\sin(\alpha-\beta)\sin(2\psi+\alpha+\beta-\gamma)$.

In addition to the cross-section and polarization, one may also obtain an experimental value for the correlation function $\mathbf{C}(\mathbf{p},\mathbf{q})=\langle\sigma\mathbf{p}\cdot\mathbf{Sq}\rangle_{\text{scatt}}$, by measuring simultaneously the polarization of the scattered beam of particles ($\mathbf{k}_f$) along a direction **p** and that of the beam of recoiling particle ($-\mathbf{k}_f$) along the direction **q**.

If the incident beam is unpolarized, one finds

$$Q^{N}C^{N}(\mathbf{N},\mathbf{N})=\frac{1}{6}\operatorname{Sp}MM^{*}(\sigma\mathbf{N})(\mathbf{SN})=\frac{2}{3}\lambda^{2}\Big\{\frac{1}{3}[|b|^{2}+|d|^{2}+|f|^{2}+|g|^{2}-|\mu|^{2}-|\nu|^{2}]+\frac{1}{\sqrt{3}}\operatorname{Re}[c^{*}f+b^{*}h-a^{*}g-d^{*}g]\Big\}, \qquad (23)$$

$$Q^{N}C^{N}(\mathbf{m},\mathbf{P})=\frac{1}{6}\operatorname{Sp}MM^{*}(\sigma\mathbf{m})(\mathbf{SP})=\frac{\lambda^{2}}{3}\{\sin(\vartheta+\beta)\cos\vartheta\,[|a|^{2}+|c|^{2}+|h|^{2}-\frac{1}{3}(|b|^{2}+|d|^{2}+|f|^{2})]$$

$$-\sin\vartheta\cos(\vartheta+\beta)\Big[\frac{2}{3}(|b|^{2}+|d|^{2}+|f|^{2})\Big]+\frac{2}{\sqrt{3}}\sin\vartheta\cos(\vartheta+\beta)\operatorname{Re}(bh^{*}+cf^{*}-ag^{*}-dg^{*})$$

$$-\frac{2}{\sqrt{3}}\cos(2\vartheta+\beta)\operatorname{Re}(ab^{*}+cd^{*}+fg^{*}+gh^{*})\},$$

β is the angle between **m** and **K**, and equals $\pi/2$ when **m** coincides with **P**. By carrying out triple scattering experiments, one may obtain additional data, for example how the direction and magnitude of polarization changes after a second scattering. In order to describe the geometry of these experiments, we introduce a unit vector **n** in the direction $[\mathbf{N}\times\mathbf{k}_i]$, lying in the plane of the second scattering. It may be seen from Eq. (19) that the cross section and polarization for second scattering may be written in the form

$$Q^{P}=Q^{N}(1+\mathbf{P}^{N}\mathbf{P}_{\text{inc}})=Q^{N}(1+A),\ \ \mathbf{P}^{P}=\mathbf{P}^{N}+\tau\mathbf{P}_{\text{inc}}/(1+\mathbf{P}^{N}\mathbf{P}_{\text{inc}}), \qquad (24)$$

where $A=\mathbf{P}^{N}\mathbf{P}_{\text{inc}}$ represents the scattering asymmetry, and τ is a second rank tensor having components

$$\tau_{ik}=\operatorname{Sp}(\sigma_{i}M\sigma_{k}M^{*})/\operatorname{Sp}(MM^{*}). \qquad (25)$$

Straightforward triple scattering experiments yield

$$Q^{N}(\mathbf{N},\tau,\mathbf{N})=\frac{1}{6}\operatorname{Sp}M\sigma_{N}M^{*}\sigma_{N}=\frac{\lambda^{2}}{3}\Big\{\frac{2}{3}\operatorname{Re}[a^{*}(d-2\mu)]+\frac{4}{3\sqrt{3}}\operatorname{Re}[(f-\nu-b)c^{*}]+\frac{8}{3\sqrt{3}}\operatorname{Re}[g^{*}(-d+\mu)]$$

$$+\frac{4}{3\sqrt{3}}[b^{*}(f-\nu)]\Big\}\qquad Q^{N}(\mathbf{m},\tau,\mathbf{n})=\frac{\lambda^{2}}{3}\sin2\vartheta\sin\vartheta\cos\beta\Big\{|a|^{2}-|h|^{2}+\frac{1}{3}(|b|^{2}+|c|^{2}-|d|^{2}+|f|^{2})-\frac{4}{3}|g|^{2}\Big\}$$

$$-\frac{2}{3}\lambda^{2}\sin2\vartheta\cos\vartheta\cos\beta\Big\{\operatorname{Re}\Big[\sqrt{\frac{1}{3}}(a^{*}b+a^{*}c)+\frac{2}{3}(c^{*}g+b^{*}g)+\frac{1}{3\sqrt{3}}(c^{*}d+b^{*}d)-\frac{4}{3\sqrt{3}}(b^{*}\mu+c^{*}\mu)\Big]\Big\}$$

$$\begin{aligned}&-\frac{2}{9}\lambda^2\sin 2\vartheta\sin\vartheta\cos\beta\Big\{\mathrm{Re}\Big[(a^*d+b^*c+f^*h)+2(a^*\mu-d^*\mu+f^*\nu+h^*\nu)+\frac{2}{\sqrt{3}}(-b^*f+b^*\nu+c^*f-c^*\nu)\\&+\frac{4}{\sqrt{3}}(d^*g-g^*\mu)\Big]\Big\}+\frac{2}{3}\lambda^2\sin(\vartheta+\beta)\Big\{\mathrm{Re}\Big[\sqrt{\frac{1}{3}}(a^*c+g^*h)+\frac{1}{3\sqrt{3}}(b^*d+g^*f)-\frac{4}{3\sqrt{3}}(b^*\mu+g^*\nu)\\&+\frac{1}{3}\Big(2c^*g+\frac{2}{3}d^*f+\frac{2}{3}d^*\nu+\frac{4}{3}f^*\mu-\frac{1}{3}\mu^*\nu\Big)\Big]\Big\}+\frac{2}{3}\lambda^2\cos(\vartheta-\beta)\Big\{\mathrm{Re}\Big[\frac{1}{3}(a^*d+b^*c+f^*h)+\frac{2}{3}(a^*\mu+h^*\nu)+\frac{2}{3\sqrt{3}}\\&\times(b^*\nu-b^*f-c^*f-c^*\nu)+\frac{4}{3\sqrt{3}}(d^*g-g^*\mu)+\frac{2}{9}(d^*\mu-f^*\nu)\Big]\Big\}+\frac{\lambda^2}{9}\cos(\vartheta-\beta)\Big\{2|g|^2+\frac{4}{3}(|d|^2-|f|^2)+\frac{1}{3}(|\mu|^2-|\nu|^2)\Big\}.\end{aligned}\tag{26}$$

Making use of the recoil beam instead of the scattered beam, we obtain two new expressions

$$\begin{aligned}&Q^{\mathrm{N}}(\mathbf{N},\tau',\mathbf{N})=\frac{1}{6}\mathrm{Sp}\,MS_NM^*\sigma_N=\frac{2}{9}\lambda^2\Big\{|g|^2+\frac{2}{3}|d|^2+|f|^2-|\mu|^2-|\nu|^2+\mathrm{Re}\,[a^*(d-\mu)-b^*(c+f+\nu)\\&+c^*\Big(\frac{2}{\sqrt{3}}f+\frac{1}{\sqrt{3}}\nu\Big)-g^*\Big(\frac{4}{\sqrt{3}}d+\frac{2}{\sqrt{3}}\mu\Big)-f^*\Big(h+\frac{1}{3}\nu\Big)+h^*\nu-d^*\mu]\Big\};\quad Q^{\mathrm{N}}(\mathbf{m},\tau',\mathbf{n})=\frac{\lambda^2}{6}\sin 2\vartheta\sin(\vartheta+\beta)\Big[2|a|^2\\&+\frac{2}{3}|b|^2+\frac{2}{3}|c|^2-\frac{2}{9}|d|^2+\frac{2}{9}|f|^2-2|g|^2-2|h|^2-\frac{2}{9}|\mu|^2+\frac{2}{9}|\nu|^2\Big]-\frac{\lambda^2}{3}\sin 2\vartheta\cos(\vartheta+\beta)\\&\times\mathrm{Re}\Big[2\sqrt{\frac{1}{3}}a^*b+\frac{4}{3}b^*g+\frac{2}{3\sqrt{3}}c^*d+\frac{4}{3\sqrt{3}}c^*\mu-\frac{1}{3\sqrt{3}}f^*g-\frac{2}{3\sqrt{3}}g^*\nu-\sqrt{\frac{1}{3}}g^*h+\frac{2}{9}f^*\mu+\frac{2}{9}d^*\nu-\frac{2}{9}d^*f\\&-\frac{2}{9}\mu^*\nu+\sqrt{\frac{1}{3}}a^*c+\frac{1}{3\sqrt{3}}b^*d+\frac{2}{3\sqrt{3}}b^*\mu+\frac{2}{3}c^*g\Big]+\frac{\lambda^2}{3}\Big[\cos(\vartheta-\beta)-\sin 2\vartheta\sin(\vartheta+\beta)\Big]2\mathrm{Re}\Big[\frac{1}{3}(a^*d-a^*\mu\\&+f^*h-h^*\nu+b^*c)+\frac{2}{3\sqrt{3}}(c^*f-b^*f+g^*\mu)+\frac{1}{3\sqrt{3}}(c^*\nu-b^*\nu+2d^*g)+\frac{1}{9}(f^*\nu-d^*\mu)\Big]+\frac{\lambda^2}{3}\sin 2\vartheta\sin(\vartheta+\beta)\frac{4}{9}\\&\times\mathrm{Re}\,[f^*\nu-d^*\mu]+\frac{\lambda^2}{3}\Big[\sin(\vartheta-\beta)-\frac{1}{2}\sin 2\vartheta\cos(\vartheta+\beta)\Big]2\mathrm{Re}\Big[\sqrt{\frac{1}{3}}(a^*c+g^*h)+\frac{1}{3\sqrt{3}}(b^*d+g^*f+2b^*\mu+2g^*\nu)+\frac{2}{3}c^*g\\&+\frac{2}{9}(d^*f+\mu^*\nu-f^*\mu-d^*\nu)\Big]+\frac{\lambda^2}{3}\Big[\cos(\vartheta-\beta)-\frac{1}{2}\sin 2\vartheta\sin(\vartheta+\beta)\Big]\Big[\frac{4}{9}(|d|^2-|f|^2)+\frac{2}{3}|g|^2-\frac{2}{9}(|\mu|^2-|\nu|^2)\Big].\end{aligned}\tag{27}$$

Simultaneous measurements of the polarizations of the scattered beam and the "recoil" beam after triple scattering, yield three further relations.

The correlation function for the polarized beam has the form

$$C^{\mathrm{P}}(\mathbf{p},\mathbf{q})=[C^{\mathrm{N}}(\mathbf{p},\mathbf{q})+\mathbf{Z}(\mathbf{p},\mathbf{q})\mathbf{P}_{\mathrm{inc}}]/(1+\mathbf{P}^{\mathrm{N}}\mathbf{P}_{\mathrm{inc}}),\tag{28}$$

where

$$C^{\mathrm{N}}(\mathbf{p},\mathbf{q})+\mathbf{Z}(\mathbf{p},\mathbf{q})\mathbf{P}_{\mathrm{inc}}=\mathrm{Sp}\,M(1+\sigma_1\mathbf{P}_{\mathrm{inc}})M^*\sigma_p S_q/\mathrm{Sp}\,MM^*,$$

and one can find

$$\begin{aligned}&Q^{\mathrm{N}}\mathbf{Z}(\mathbf{m},\mathbf{P})\,\mathbf{N}=\frac{1}{6}\mathrm{Sp}\,M\sigma_NM^*\sigma_m S_p=\frac{2}{3}\lambda^2\,\mathrm{Im}\Big\{\sin(\vartheta+\beta)\cos\vartheta\Big[\sqrt{\frac{1}{3}}(ac+gh^*)-\frac{2}{3\sqrt{3}}(b\mu^*-g\nu^*)+\frac{2}{3}cg^*\\&+\frac{2}{9}(d\nu^*-f\mu^*-df^*)-\frac{1}{3\sqrt{3}}(bd^*+fg^*)\Big]+\cos(\vartheta+\beta)\sin\Big[-\frac{1}{3\sqrt{3}}(b\mu^*+c\mu^*)+\frac{1}{3}(cg^*+af^*-a\nu^*-bg^*-dh^*-h\mu^*)\\&+\frac{1}{9}(d\nu^*-f\mu^*-4df^*)-\frac{2}{3\sqrt{3}}(bd^*+cd^*)\Big]+\cos(\vartheta+\beta)\cos\vartheta\Big[-\frac{1}{3}(a\mu^*-h\nu^*+ad^*+bc^*+fh^*)+\frac{1}{3\sqrt{3}}(c\nu^*-g\mu^*)\\&-\frac{2}{3\sqrt{3}}(cf^*+dg^*)\Big]+\sin(\vartheta+\beta)\sin\vartheta\Big[\frac{1}{3}(-g\mu^*+ad^*+bc^*+fh^*-d\mu^*-f\nu^*+bc^*)+\frac{2}{3\sqrt{3}}(cf^*+dg^*)-\sqrt{\frac{1}{3}}b\nu^*\Big]\\&+\cos\beta\Big[-\frac{1}{3}(a\mu^*-h\nu^*)+\frac{1}{3\sqrt{3}}(c\nu^*-g\mu^*)\Big]-\frac{2}{9}\mu\nu^*\sin\beta\Big\},\ Q^{\mathrm{N}}\mathbf{Z}(\mathbf{N},\mathbf{P})\,\mathbf{n}=\frac{2}{3}\lambda^2\,\mathrm{Im}\Big\{\frac{1}{2}\sin 2\vartheta\sqrt{\frac{1}{3}}(ba^*+b\mu^*+g\nu^*\\&-gh^*-ag^*)+\frac{1}{3}(-af^*+a\nu^*+bg^*+cg^*-dh^*-d\nu^*-f\mu^*-h\mu^*-2a\mu^*+bc^*+2d\mu^*+fh^*-2f\nu^*-2h\nu^*)\\&+\frac{1}{3\sqrt{3}}(c\mu^*-cd^*-fg^*-g\nu^*-cf^*+dg^*)+\frac{4}{3\sqrt{3}}(c\mu^*-g\nu^*+c\nu^*+g\mu^*)+\frac{2}{3\sqrt{3}}fg^*\Big]+\sin^2\vartheta\Big[\sqrt{\frac{1}{3}}(-ag^*+bh^*\\&-b\mu^*-g\nu^*)+\frac{1}{3}(af^*-a\nu^*-bg^*-cg^*+dh^*+d\nu^*+f\mu^*+h\mu^*)+\frac{1}{3\sqrt{3}}(cf^*-dg^*-c\mu^*+g\nu^*)+\frac{2}{3\sqrt{3}}(c\nu^*+g\mu^*\\&-cd^*-fg^*)+\frac{2}{3}(d\mu^*-f\nu^*)\Big]+\cos^2\vartheta\Big[\frac{1}{3}(ad^*-bc^*-fh^*)+\frac{2}{3}(a\mu^*+h\nu^*)+\frac{2}{3\sqrt{3}}(cf^*-c\nu^*-dg^*-g\mu^*)\end{aligned}$$

$$-\sqrt{\frac{1}{3}}(ab^* + gh^*) - \frac{1}{3\sqrt{3}}(cd^* + fg^*) + \frac{4}{3\sqrt{3}}(c\mu^* - g\nu^*)\Big]\Big\}, \quad Q^N\mathbf{Z}(\mathbf{m}, \mathbf{N})\,\mathbf{n} = \frac{\lambda^2}{3}\sin 2\vartheta \sin(\vartheta+\beta)\,\mathrm{Im}\Big\{-\frac{2}{\sqrt{3}}(ab^* + gh^*)$$

$$-\sqrt{\frac{1}{3}}(b\mu^* + c\mu^*) + \frac{1}{3}(-af^* + a\nu^* + bg^* + cg^* - dh^* - h\mu^*) - (d\nu^* + f\mu^*)\Big\} + \frac{\lambda^2}{3}\sin 2\vartheta\cos(\vartheta+\beta)\,\mathrm{Im}\Big\{\sqrt{\frac{1}{3}}(2ag^*$$

$$-2bh^* - b\nu^* - c\nu^*) + \frac{1}{3}(ad^* - a\mu^* - bc^* - h\nu^* - fh^*) - f\nu^* + d\mu^*\Big\} - \lambda^2\Big[\frac{2}{3}\cos(\vartheta-\beta) - \frac{1}{2}\sin 2\vartheta\sin(\vartheta+\beta)\Big]$$

$$\times\,\mathrm{Im}\Big[\frac{1}{3}(-af^* + a\nu^* + bg^* + cg^* - dh^* + d\nu^* + f\mu^* - h\mu^*) + \frac{1}{3\sqrt{3}}(c\mu^* + 2cd^* + 2fg^* - 4g\nu^*) - \sqrt{\frac{1}{3}}b\mu^*\Big]$$

$$+\lambda^2\Big[\frac{2}{3}\sin(\vartheta-\beta) - \frac{1}{3}\sin 2\vartheta\cos(\vartheta+\beta)\Big]\mathrm{Im}\Big[\frac{1}{3}(-ad^* + a\mu^* + bc^* + d\mu^* \tag{29}$$

$$+fh^* - f\nu^* + h\nu^*) - \frac{1}{3\sqrt{3}}(c\nu^* - 2dg^* + 2cf^* + 4g\mu^*) + \sqrt{\frac{1}{3}}b\nu^*\Big].$$

These same results can be obtained by using the expression for M given by Eq. (11). In this case, one finds, for example,

$$Q^N = A^2 + \frac{2}{3}B^2 + \frac{2}{9}C^2 + A_1^2 + \frac{2}{3}B_1^2 + \frac{2}{9}C_1^2 + \frac{4}{3}D^2 + \frac{4}{3}E^2 + \frac{1}{6}F^2 + \frac{1}{6}G^2 + \frac{1}{6}H^2 + \frac{1}{6}K^2,$$

$$Q^N P^N = 2\mathrm{Re}\Big\{AA_1^* + \frac{2}{3}BB_1^* + \frac{2}{9}CC_1^*\Big\},$$

$$Q^N(\mathbf{N},\tau,\mathbf{N}) = |A|^2 + \frac{2}{3}|B|^2 + \frac{2}{9}|C|^2 + |A_1|^2 + \frac{2}{3}|B_1|^2 + \frac{2}{9}|C_1|^2 \tag{30}$$

$$-\frac{4}{3}(|D|^2 + |E|^2) - \frac{1}{6}(|F|^2 + |G|^2 + |H|^2 + |K|^2).$$

The coefficients A, B, C, ..., are easily obtained from Eq. (11), substituting in it the explicit form of M given in (14) and the average values of the spin operator expressed interms of Eulerian angles; one finds

$$A = \frac{1}{6}\mathrm{Sp}\,M = \frac{\lambda}{3}(a + d + \mu), \qquad A_1 = \frac{1}{6}\mathrm{Sp}\,M\sigma\mathbf{N} =$$

$$B = \frac{1}{4}\mathrm{Sp}\,MSN = \qquad = \lambda\frac{i}{3}\Big[\sqrt{\frac{1}{3}}(c-b) + \frac{2}{3}f + \frac{1}{3}\nu\Big],$$

$$= \frac{i}{2}\lambda\Big[\sqrt{\frac{1}{3}}(c-b) + \frac{2}{3}(f-\nu)\Big], \qquad B_1 = \frac{\lambda}{3}[(d-\mu) - \sqrt{3}g],$$

$$C = \frac{3}{4}\mathrm{Sp}M\Big(S_PS_P + S_KS_K - \frac{4}{3}\delta_{ij}\Big) = \qquad C_1 = \frac{3}{4}\mathrm{Sp}M\sigma\mathbf{N}\Big(S_PS_P + S_KS_K - \frac{4}{3}\delta_{ik}\Big) =$$

$$= \frac{3}{4}\lambda\Big[\frac{1}{3}(a-d) + \frac{2}{\sqrt{3}}g\Big], \qquad = \frac{3}{4}i\lambda\Big[\frac{1}{3\sqrt{3}}(c-b) - \frac{7}{9}f + h - \frac{8}{9}\nu\Big],$$

$$D = \frac{1}{8}\mathrm{Sp}\,M(\sigma_PS_P + \sigma_KS_K) = \frac{\lambda}{4}\Big(a + \frac{1}{3}d + \frac{2}{\sqrt{3}}g - \frac{4}{3}\mu\Big),$$

$$E = \frac{1}{8}\mathrm{Sp}\,M(\sigma_PS_P - \sigma_KS_K) = \frac{\lambda}{4}\Big\{\Big(a - d - \frac{2}{\sqrt{3}}g\Big)\cos 2\vartheta + \frac{2}{\sqrt{3}}(b+c)\sin 2\vartheta\Big\} = \frac{\lambda}{4}\frac{1}{\sin 2\vartheta}\frac{2}{\sqrt{3}}(b+c), \tag{31}$$

$$F = \frac{1}{2}\mathrm{Sp}\,M(\sigma_PS_PS_N + \sigma_KS_KS_N) = \frac{\lambda i}{\sqrt{2}}\Big\{\sqrt{\frac{1}{6}}(c-b) - \frac{1}{3\sqrt{2}}f + \sqrt{\frac{1}{2}}h + \frac{2\sqrt{2}}{3}\nu\Big\},$$

$$G = \frac{1}{2}\mathrm{Sp}\,M(\sigma_PS_NS_P + \sigma_KS_NS_K) = \frac{\lambda i}{\sqrt{2}}\Big\{\sqrt{\frac{1}{6}}(c-b) - \frac{1}{3\sqrt{2}}f + \sqrt{\frac{1}{2}}h + \frac{2\sqrt{2}}{3}\nu\Big\},$$

$$H = \frac{1}{2}\mathrm{Sp}\,M(\sigma_PS_PS_N - \sigma_KS_KS_N) = \frac{\lambda i}{\sqrt{2}}\Big\{\sin 2\vartheta\Big(\sqrt{\frac{1}{2}}a - \sqrt{\frac{2}{3}}g - \sqrt{\frac{1}{2}}d\Big) + \cos 2\vartheta\Big[-\sqrt{\frac{3}{2}}b - \sqrt{\frac{1}{6}}c - \sqrt{\frac{1}{2}}(f+h)\Big]\Big\}.$$

$$K = \frac{1}{2}\mathrm{Sp}\,M(\sigma_PS_NS_P - \sigma_KS_NS_K) = \frac{\lambda i}{\sqrt{2}}\Big\{\sin 2\vartheta\Big(-\sqrt{\frac{1}{2}}a + \sqrt{\frac{2}{3}}g + \sqrt{\frac{1}{2}}d\Big) + \cos 2\vartheta\Big[\sqrt{\frac{3}{2}}c + \sqrt{\frac{1}{6}}b - \sqrt{\frac{1}{2}}(f+h)\Big]\Big\}.$$

With these expressions for the coefficients, one can use Eq. (30) to obtain the previous results for the cross section, polarization and $Q^N(\mathbf{N}, \tau, \mathbf{N})$.

Thus, for given values of energy and angle, the results of double and triple scattering experiments can yield 11 independent relations among the parameters of the transition matrix in the case of elastic scattering of neutrons against deuterons. One additional relation can be obtained from experiments on the scattering of protons against deuterons, since due to charge invariance, all the above results also apply to this case (if one only includes the electromagnetic interaction).

As shown in the work of Smorodinskii and others,[5] the transition matrix is determined by as many real functions as there are variables in its most general formulation. Thus, if one carries out triple scattering experiments and obtains the above mentioned experimental data, it should be possible to carry out a phase shift analysis.

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NUCLEAR FORCES AND THE SCATTERING OF π MESONS

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The interaction of two fixed nucleons in pseudoscalar meson theory with pseudovector coupling is considered. The principal part of the functional of the two-nucleon state is represented in the form of a product of single-nucleon functionals. Consideration is given only to the states without real mesons and with one real meson. A procedure is developed for reducing two-nucleon renormalized matrix elements to single nucleon elements, which are then calculated by the method of Chew, Low, and Wick. The potential of order e^{-2R} is calculated. It consists of two parts: one part is proportional to f^4 (f is the interaction constant), and the other is a function of the phases of π meson scattering on nucleons.

RECENTLY, Chew and Low[1] and Wick[2] considered the one-nucleon problem from a new point of view. Characteristic of their approach is the attempt to solve the problem without perturbation theory, and thus to deal only with renormalized quantities.

In considering the two-nucleon problem in the region of nonrelativistic energies, it may be assumed that the meson clouds of the interacting nucleons conserve their individuality.

Therefore, we may feel confident that in this energy region, quantities referring to two interacting nucleons will be expressed by single-nucleon quantities, so that the method of Chew, Low, and Wick may also be used in the two-nucleon problem. In the present work, we will approach the theory of nucleon interaction from this point of view.*

1. STATEMENT OF THE PROBLEM

We consider a field of π mesons, which interacts with two nucleons, fixed at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. The energy operator for such a system is

$$H = H_0 + U_1 + U_2, \tag{1}$$

where H_0 is the energy operator of the meson field

$$H_0 = \sum_q q_0 a_q^+ a_q, \tag{2}$$

and U_A is the interaction operator of nucleon A with the meson field in symmetrical theory

$$U_A = \sum_q \{V_{Aq}^0 a_q e^{i\mathbf{q}\mathbf{r}_A} + V_{Aq}^0 a_q^+ e^{-i\mathbf{q}\mathbf{r}_A}\}; \tag{3}$$

$$V_{Aq}^0 = if_0 [(\sigma_A \mathbf{q}) / \sqrt{2q_0}] \tau_{Aq} v(q). \tag{4}$$

Here, $q_0 = (1 + q^2)^{1/2}$, the spin operator σ_A and the isotopic spin operator τ_A refer to nucleon A; a_q and a_q^+ are the annihilation and creation operators of a meson in the state q; for simplicity, one symbol q is used to denote all the quantum numbers of the meson — the momentum and the third projection of the isotopic spin. We use a system of units in which $\hbar = c = \mu = 1$ (μ is the meson mass), f_0 is the non-renormalized coupling constant (in a rationalized system), and v(q) is the source function.

The eigenfunctions Ψ^n of the Hamiltonian H refer to states with two nucleons and different numbers of real mesons: without mesons (n = 0), with one meson (n = 1), etc. We are interested in Ψ^0 — the state with two nucleons, fixed at a distance $R = |\mathbf{r}_2 - \mathbf{r}_1|$, and surrounded by a cloud of virtual mesons. In the representation in which the creation operator a_q^+ is a multiplicative operator on an auxiliary function $\bar{a}_q$, the Schroedinger equation for Ψ^0 is written in the form

$$H\Psi_\sigma^0(1, 2, \bar{a}) = \{2E_0 + E_\sigma(\mathbf{R})\}\Psi_\sigma^0(1, 2, \bar{a}). \tag{5}$$

The energy eigenvalue consists of two parts — the self-energy of both nucleons $2E_0$ and the static interaction energy of the nucleons $E_\sigma(R)$; $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. The index $\sigma \equiv (S', I', S_3, I_3)$ characterizes the eigenvalues of the total spin, total isotopic spin, and their third projections. The state vector $\Psi^0(1, 2, \bar{a})$ is a function of the parameters $\mathbf{r}_1$, $\mathbf{r}_2$ and the spin-charge variables of nucleons 1 and 2. Moreover, $\Psi_\sigma^0(1, 2, \bar{a})$ is a function of the variables of the virtual mesons, and Ψ_σ^0 is a functional of $\bar{a}_q$. It is sometimes convenient to use the representation

$$\Psi_\sigma^0(1, 2, \bar{a}) = \Psi_\sigma^0(1, 2, a^+)\Lambda_0, \tag{6}$$

where Λ_0 is the vacuum state of the meson field.

Our problem is the calculation of Ψ_σ^0 and $E_\sigma(\mathbf{R})$. We will attempt to reduce the two-nucleon problem to a one-nucleon problem. For this purpose it is necessary to study products of one-nucleon state vectors $F_\alpha(1, \bar{a})$ and $F_\beta(2, \bar{a})$ (α is the spin-charge index, taking four values). The quantity $F(1, \bar{a})$ is a solution of the Schrödinger equation

$$H_1 F(1, \bar{a}) \equiv (H_0 + U_1) F(1, \bar{a}) = E_0 F(1, \bar{a}). \tag{7}$$

Similar to $\Psi^0(1, 2, \bar{a})$, the one-nucleon state $F(1, \bar{a})$ is also a function of the variables of the meson field. We will use the formula

$$F(1, \bar{a}) = F(1, a^+)\Lambda_0. \tag{8}$$

Different spin-charge states are orthogonal

$$(F_\alpha(1, \bar{a}), F_\beta(1, \bar{a})) = \delta_{\alpha\beta}. \tag{9}$$

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*A brief communication of part of the results was published in J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1262 (1957) [Soviet Phys. JETP 5, 1030 (1957)].

The dependence of $F(1, \bar{a})$ on the coordinate $\mathbf{r}_1$ has the form

$$F(1, \bar{a}) = e^{-i\mathbf{u}\mathbf{r}_1} F^0(1, \bar{a}) = F(1, \bar{a}_q e^{-i\mathbf{q}\mathbf{r}_1}), \tag{10}$$

where $\mathbf{u}$ is the momentum operator of the meson field, and $F^0(1, \bar{a})$ is independent of $\mathbf{r}_1$.

We will consider the asymptotic behavior of $\Psi^0(1, 2, \bar{a})$ at great distances R, at which the nucleons do not interact. In this case the meson cloud in each nucleon will have the same character as in the absence of the other nucleon. Therefore, as $R \to \infty$, the functional of the two-nucleon state $\Psi^0(1, 2, \bar{a})$ must be a linear combination of products of the one-nucleon functionals $F(1, \bar{a}) F(2, \bar{a})$: as $R \to \infty$

$$\Psi^0_\sigma(1, 2, \bar{a}) \to \Phi^\sigma(1, 2, \bar{a}) = \sum c_{\alpha\beta} F_\alpha(1, \bar{a}) F_\beta(2, \bar{a}), \tag{11}$$

i.e., the functional $\Phi^\sigma(1, 2, \bar{a})$ must be a solution of the Schrödinger equation

$$(H - 2E_0)\Phi^\sigma(1, 2, \bar{a}) = 0, \quad R \to \infty. \tag{12}$$

This was shown by Ekstein[3] in a nonstationary treatment. For the stationary treatment adopted by us, the asymptotic behavior of Ψ^0 and the state vector with one real meson Ψ^q are considered in the Appendix.

The description of noninteracting nucleons by the use of the state vector (11) differs from the usual description in that the meson clouds of both nucleons are taken into account from the very beginning. The products $\Phi_{\alpha\beta}(1, 2, \bar{a}) = F_\alpha(1, \bar{a}) F_\beta(2, \bar{a})$ are orthogonal only as $R \to \infty$

$$(\Phi_{\alpha\beta}(1, 2, \bar{a}), \Phi_{\alpha'\beta'}(1, 2, \bar{a})) = \delta_{\alpha\alpha'}\delta_{\beta\beta'}, \quad R \to \infty. \tag{13}$$

For finite R, a term depending on R and not containing $\delta_{\alpha\alpha'}\delta_{\beta\beta'}$ is added to the right hand side of Eq. (13). The nonorthogonality of $\Phi_{\alpha\beta}(1, 2, \bar{a})$ is a result of the fact that the kinematic interaction, which is associated with the identity and statistics of the mesons and is similar to the interaction by means of Pauli's principle for a Fermi field, is already taken into account in $\Phi_{\alpha\beta}$. The region of nonorthogonality may be identified with the region in which the meson clouds overlap, and one and the same meson may belong to both clouds.

We will seek a solution of the Schrödinger equation (5) for finite R in the form:

$$\Psi^0_\sigma(1, 2, \bar{a}) = \Phi^\sigma(1, 2, \bar{a}) + \chi_\sigma(1, 2, \bar{a}), \tag{14}$$

where $\Phi^\sigma(1, 2, \bar{a})$, defined by Eq. (11), coincides with $\Psi^0_\sigma(1, 2, \bar{a})$ as $R \to \infty$. Only the states Ψ^0_μ and Ψ^0_μ (with one real meson) are considered in the expansion of χ_σ in eigenfunctions of the operator H. We are interested only in distances at which potentials of the order e^{-R} and e^{-2R} are of fundamental importance. As we will see below, in Sec. 3, the first term of Φ^σ in Eq. (14) already yields the potential e^{-R} and the major part of the potential e^{-2R}. We therefore consider χ_σ to be a small quantity. The terms neglected here yield terms in the potential of the order e^{-3R} and higher. We then obtain

$$\Psi^0_\sigma(1, 2, \bar{a}) = \frac{1}{(\Phi^\sigma, \Phi^\sigma)}\left\{\Phi^\sigma - \sum_{\mu \neq \sigma}(\Phi^\mu, \Phi^\sigma)\Phi^\mu - \sum_{\mu q}\frac{1}{q_0}(\Phi^{q\mu}, H\Phi^\sigma)\Phi^{q\mu}\right\}; \tag{15}$$

$$E_\sigma(R) = \frac{1}{(\Phi^\sigma, \Phi^\sigma)}\left\{(\Phi^\sigma, H\Phi^\sigma) - \sum_{\mu \neq \sigma}(\Phi^\mu, \Phi^\sigma)(\Phi^\sigma, H\Phi^\mu)\right.$$

$$\left. - \sum_{\mu q}\frac{1}{q_0}(\Phi^{q\mu}, H\Phi^\sigma)(\Phi^\sigma, H\Phi^{q\mu})\right\}. \tag{16}$$

Here and in the future, we will assume $E_0 = 0$, since the self-energy enters everywhere only as a difference. We replaced Ψ^0_σ by Φ^σ in the denominators of Eqs. (15) and (16), which corresponds to taking account of terms of the order e^{-R}. Let us note that the smallness of χ_σ is not associated with the smallness of the coupling constant. $\Phi^{q\mu}$ is defined by Eq. (A.6).

2. INTRODUCTION OF SEPARATE COORDINATES FOR MESON CLOUDS

The matrix elements in Eqs. (15) and (16) have the form

$$(\alpha\beta | L | \alpha'\beta') = (F_\alpha(1, \bar{a}) F_\beta(2, \bar{a}), L(a, a^+) F_{\alpha'}(1, \bar{a}) F_{\beta'}(2, \bar{a})), \tag{17}$$

where L contains the normal product of the creation and annihilation operators, and is a function of the operators σ_A and τ_A. Such expressions may be calculated if the one-nucleon functionals $F(1, \bar{a})$ and $F(2, \bar{a})$ (which may be found, for example, by the use of the intermediate coupling method) are known. However, the use of an explicit form of the one-nucleon functionals is undesirable because it would lead to the consideration of virtual mesons belonging to the clouds, and the effect of the self-energy would be mixed with the effects of the interaction. Therefore, we will attempt to calculate matrix elements of the type (17) by expressing them through matrix elements between states of the physical nucleon. This will allow us to use only the general properties of the functionals $F(1, \bar{a})$ and $F(2, \bar{a})$, and not their concrete form. With this approach, all the matrix elements encountered will be renormalized.

In the coordinate representation, the roles of annihilation and creation operators are fulfilled by $\varphi^{(+)}(\mathbf{r})$ and $\varphi^{(-)}(\mathbf{r})$ — the annihilating and creating parts of the meson operator $\varphi(\mathbf{r})$. If $\varphi^{(-)}(\mathbf{r})$ is a multiplicative operator, then the operator $\varphi^{(+)}(\mathbf{r})$ has the form:

$$\varphi^{(+)}(\mathbf{r}) = \int \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \frac{\delta}{\delta\varphi^{(-)}(\mathbf{r}')}, \quad \Delta^{(+)}(\mathbf{r}-\mathbf{r}') = [\varphi^{(+)}(\mathbf{r}), \varphi^{(-)}(\mathbf{r}')]. \tag{18}$$

Let B_A be the region occupied by the meson cloud of nucleon A. Then $F(A, a^+)$ depends only on those variables of $\varphi^{(-)}(\mathbf{r})$ for which $\mathbf{r}$ lies in the region $B_A: F(A, a^+) = F(A, \varphi^{(-)}_{B_A})$.

We will introduce separate variables for the meson clouds of nucleons 1 and 2. We will denote the field variables in the functional $F(1, \varphi^{(-)}_{B_1})$ by $\varphi_1^{(-)}(\mathbf{r})$, and the field variables in the functional $F(2, \varphi^{(-)}_{B_2})$ by $\varphi_2^{(-)}(\mathbf{r})$. We will introduce the annihilation operators $\varphi_1^{(+)}(\mathbf{r})$ and $\varphi_2^{(+)}(\mathbf{r})$ and with the commutators

$$[\varphi_1^{(+)}(\mathbf{r}), \varphi_1^{(-)}(\mathbf{r}')] = [\varphi_2^{(+)}(\mathbf{r}), \varphi_2^{(-)}(\mathbf{r}')] = \Delta^{(+)}(\mathbf{r}-\mathbf{r}'), \ [\varphi_1^{(+)}(\mathbf{r}), \varphi_2^{(-)}(\mathbf{r}')] = 0. \tag{19}$$

The operator $\varphi_A^{(+)}(\mathbf{r})$ has the meaning of an annihilation operator of a meson of the cloud of nucleon A. With regard to the product $F(1, \varphi_1^{(-)})\, F(2, \varphi_2^{(-)})$, the operator $\varphi^{(+)}(\mathbf{r})$ will be equivalent to the operator

$$\varphi^{(+)}(\mathbf{r}) = \varphi_1^{(+)}(\mathbf{r}) + \varphi_2^{(+)}(\mathbf{r}) = \int \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \left\{\frac{\delta}{\delta\varphi_1^{(-)}(\mathbf{r}')} + \frac{\delta}{\delta\varphi_2^{(-)}(\mathbf{r}')}\right\}. \tag{20}$$

For L = 1, the matrix element (17) is now written in the form:

$$(\Lambda_0, F^*(1, \varphi^{(+)})\, F^*(2, \varphi^{(+)})\, F(1, \varphi_1^{(-)})\, F(2, \varphi_2^{(-)})\, \Lambda_0). \tag{21}$$

[No assumptions are made in going from Eq. (17) to Eq. (21)]. It is clear from Eqs. (20) and (21) that the introduction of separate variables for the meson clouds will be convenient in the case in which $F^*(A, \varphi^{(+)}) = F^*(A, \varphi_1^{(+)} + \varphi_2^{(+)})$ may be expressed by $F^*(A, \varphi_A^{(+)})$.

We will assume that the commutator $[\varphi_1^{(+)}(\mathbf{r}_2), F(1, \varphi_1^{(-)})]$ is considerably smaller than the commutator $[\varphi_2^{(+)}(\mathbf{r}_2), F(2, \varphi_2^{(-)})]$. This corresponds to the assumption that in the common meson cloud of two interacting nucleons, the meson clouds of the individual mesons may be distinguished. Then in the calculation of (21), we may consider the operator $\varphi_2^{(+)}$ in $F^*(1, \varphi_1^{(+)})$ to be small compared with $\varphi_1^{(+)}$, and the operator $\varphi_1^{(+)}$ in $F^*(2, \varphi_1^{(+)} + \varphi_2^{(+)})$ to be small compared with $\varphi_2^{(+)}$. For small $\varphi_2^{(+)}$

$$F^*(1, \varphi_1^{(+)} + \varphi_2^{(+)}) \approx F^*(1, \varphi_1^{(+)}) + \int \frac{\delta F^*(1, \varphi_1^{(+)})}{\delta\varphi_1^{(+)}(\mathbf{r})} d^3r\, \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \frac{\delta}{\delta\varphi_2^{(-)}(\mathbf{r}')} + \cdots \tag{22}$$

and similarly for $F^*(2, \varphi_1^{(+)} + \varphi_2^{(+)})$. Making the transition to the creation and annihilation operators a^+_{Aq} and a_{Aq} of a meson in the A th cloud in momentum space, we obtain

$$(\alpha\beta \,|\, \alpha'\beta') = (F_\alpha(1, \bar{a}_1)\, F_\beta(2, \bar{a}_2), \ (1 + \hat{N})\, F_{\alpha'}(1, \bar{a}_1)\, F_{\beta'}(2, \bar{a}_2)), \tag{23}$$

$$\hat{N} = \sum_q [a^+_{1q} a_{2q} + a^+_{2q} a_{1q}]. \tag{24}$$

In the general case, in the calculation of (17), we must first apply all the operators a_q and a_q^+ to the functionals $F(1, \bar{a})$, $F(2, \bar{a})$, and then use expansions of the type (22).

The first term in (22) is the functional of nucleon 1, not interacting with nucleon 2. The second term in (22), which decreases with the distance R as e^{-R}, is associated with the exchange of mesons between clouds 1 and 2. The following term in expansion (22) decreases as e^{-2R}. Retaining only the first term in expansions of the type (22), we obtain a potential of the order of e^{-R}. If we also include the second term, we find a potential of the order of e^{-2R}.

As a result, we arrive at matrix elements of products of the operators a_{Aq} and $\overset{+}{a}_{Aq}$ between states of the noninteracting physical nucleons $F(1, \bar{a}_1)\, F(2, \bar{a}_2)$. These matrix elements in turn reduce to products of one-nucleon matrix elements.

From the products $\Phi_{\alpha\beta}(1, 2; \bar{a}_1, \bar{a}_2) = F_\alpha(1, \bar{a}_1)\, F(2, \bar{a}_2)$ we may already construct in the usual way the eigenfunctions $\Phi^\sigma(1, 2; \bar{a}_1, \bar{a}_2)$, $\sigma \equiv (S', I', S_3', I_3')$. The spin and isotopic spin operators of two noninteracting nucleons are defined as $S = S_1(a_1) + S_2(a_2)$; $I = I_1(a_1) + I_2(a_2)$, where S_A and I_A refer to an isolated nucleon with a meson cloud.

3. CALCULATION OF THE TWO-NUCLEON POTENTIAL

In the future we will write $|\alpha\beta>$ instead of the state $F_\alpha(1, \bar{a}_1)\, F_\beta(2, \bar{a}_2)$ and $|\alpha>$ instead of the one-nucleon state $F_\alpha(1, \bar{a}_1)$. In the calculations we will frequently use the following formulas from the works of Chew and Low[1] and of Wick[2]:

$$a_{1q}|\alpha\rangle = -\frac{1}{H_1 + q_0} V_{1q}^0 e^{-i\mathbf{q}\mathbf{r}_1} |\alpha\rangle, \tag{25}$$

$$\langle\alpha|V_q^0|\alpha'\rangle = (u_\alpha V_q u_{\alpha'}), \tag{26}$$

where $H_1 = H_0(a_1) + U_1(a_1)$; u_α is the spin-charge function of the bare nucleon, and V_q contains the renormalized charge f.

We find from Eqs. (23) and (24) and the normalization condition (9),

$$\begin{aligned}(\alpha\beta|\alpha'\beta') = \langle\alpha\beta|1 + \hat{N}|\alpha'\beta'\rangle = \delta_{\alpha\alpha'}\delta_{\beta\beta'} + \sum_q \{\langle\alpha|a_{1q}^+|\alpha'\rangle\langle\beta|a_{2q}|\beta'\rangle + \langle\alpha|a_{1q}|\alpha'\rangle\langle\beta|a_{2q}^+|\beta'\rangle\} \\ = \delta_{\alpha\alpha'}\delta_{\beta\beta'} + (u_\alpha(1)\, u_\beta(2), N u_{\alpha'}(1)\, u_{\beta'}(2)),\end{aligned} \tag{27}$$

where N depends only on the spin and charge operators:

$$N = \frac{f^2}{(2\pi)^3} \int \frac{d^3q}{q_0^3} e^{i\mathbf{q}\mathbf{R}} (\boldsymbol{\sigma}_1\mathbf{q})(\boldsymbol{\sigma}_2\mathbf{q})(\tau_1\tau_2)\, v^2(q). \tag{28}$$

It follows from the form of the operator N that the functionals $\Phi^\sigma(1, 2, \bar{a})$ are orthogonal:

$$(\Phi^\sigma(1, 2, \bar{a}),\ \Phi^{\sigma'}(1, 2, \bar{a})) = \delta_{\sigma\sigma'}(1 + N_\sigma), \quad \sigma \equiv (S', I', S'_3, I_3'), \tag{29}$$

where N_σ is the mean value of N in the state σ.

The potential $W(R)$ between the nucleons is defined as an operator whose mean value in the state σ (in a phenomenological treatment) is equal to the interaction energy E_σ, calculated by Eq. (16). It follows from Eq. (29) that the second term in Eq. (16) is zero. We will consider the principal term in the interaction energy (26), which is equal to

$$(\Phi^\sigma,\ H\Phi^\sigma)/(1 + N_\sigma). \tag{30}$$

The numerator in Eq. (30) is a linear combination of matrix elements $(\alpha\beta|H|\alpha'\beta')$, which by the use of Eqs (A.3) and (22) may be reduced to the form

$$(\alpha\beta|H|\alpha'\beta') = \langle\alpha\beta|(1 + \hat{N})\,[U_1^+(a_2) + U_2^+(a_1)]\,|\alpha'\beta'\rangle. \tag{31}$$

Here, $U_1^+(a_2)$ is the annihilating part of the operator U_1 with annihilation operators a_{2q}. If we neglect the weighting operator $\hat{N}$ in Eq. (31), i.e., we neglect the distortion of the meson clouds, then by using (25) and (26) we obtain the usual pseudovector interaction in the lowest approximation of perturbation theory (with renormalized charge f):

$$\langle\alpha\beta\,|\,U_1^+(a_2)+U_2^+(a_1)\,|\,\alpha'\beta'\rangle=(u_\alpha(1)\,u_\beta(2),\ W_0u_{\alpha'}(1)\,u_{\beta'}(2)),\tag{32}$$

where W_0 is a pseudovector potential

$$W_0(R)=-\frac{f^2}{(2\pi)^3}\int\frac{d^3q}{q_0^2}\,e^{i\mathbf{qR}}(\sigma_1\mathbf{q})(\sigma_2\mathbf{q})(\tau_1\tau_2)\,v^2(q).\tag{33}$$

The second part $(\alpha\beta|\,H|\,\alpha'\beta')$, which depends on the operator $\hat{N}$, leads to an interaction energy of the order e^{-2R}. This part contains products of matrix elements of the first nucleon

$$\langle\alpha\,|\,a_{1q}^+V_{1q'}^0\,|\,\alpha'\rangle,\quad\langle\alpha\,|\,a_{1q}a_{1q'}\,|\,\alpha'\rangle,\quad\langle\alpha\,|\,a_{1q}^+a_{1q'}\,|\,\alpha'\rangle$$

times analogous matrix elements of the second nucleon. Each of these matrix elements may be calculated by the methods of Chew and Low[1] and of Wick.[2] For example

$$\langle\alpha\,|\,a_{1q}^+V_{1q'}^0\,|\,\alpha'\rangle=-\sum_{n_1}\frac{\langle\alpha\,|\,V_{1q}^0\,|\,n_1\rangle\langle n_1\,|\,V_{1q'}^0\,|\,\alpha'\rangle}{E_{n_1}+q_0},\tag{34}$$

where $|n_i\rangle$ is a complete set of eigenfunctionals of the one-nucleon energy operator H_1. In the product of the sum (34) times the analogous sum for nucleon 2 (with summation index n_2), we must leave only the terms $n_1=n_2=0$ (no mesons; functional $|\mu\rangle$ and $|\nu\rangle$), and also $n_1=1$, $n_2=0$ and $n_1=0$, $n_2=1$ (one meson; functionals $|q\mu\rangle$ and $|q\nu\rangle$), because we are considering only states without mesons and with one real meson.

We will first single out the term $n_1=n_2=0$ from $\langle\alpha\beta|\,\hat{N}\,[U_1^+(a_2)+U_2^+(a_1)]\,|\,\alpha'\beta'\rangle$. After some transformations, it has the form:

$$-\sum_{qq'}e^{i(\mathbf{q}+\mathbf{q}')\mathbf{R}}\Big(u_\alpha(1)\,u_\beta(2),\,2V_{1q}V_{1q'}V_{2q}V_{2q'}\,\frac{q_0+q_0'}{q_0^2\,q_0'^2}\,u_{\alpha'}(1)\,u_{\beta'}(2)\Big)+(u_\alpha(1)\,u_\beta(2),\ W_1(\mathrm{R})\,u_{\alpha'}(1)\,u_{\beta'}(2)).\tag{35}$$

In the calculation of $(\Phi^\sigma,\,H\Phi^\sigma)$, the first term in (35) gives $N_\sigma W_{0\sigma}$, where $W_{0\sigma}$ is the mean value of the potential (33) in the state σ. The second term in (35) contributes $W_{1\sigma}/(1+N_\sigma)\approx W_{1\sigma}$ to the energy (30), if the interaction of order e^{-3R} is neglected. Thus, the without-mesons part of the matrix element $(\alpha\beta|\,H|\,\alpha'\beta')$ leads to the potential W_0+W_1, where

$$W_1(\mathrm{R})=-\frac{f^4}{(2\pi)^6}\int d^3q\,d^3q'\,e^{i(\mathbf{q}+\mathbf{q}')\mathbf{R}}\,\frac{(2q_0+q_0')\,v^2(q)\,v^2(q')}{q_0^2q_0'^3(q_0+q_0')}\,\{2(\tau_1\tau_2)(\mathbf{qq}')^2+3(\sigma_1[\mathbf{qq}'])(\sigma_2[\mathbf{qq}'])\}.\tag{36}$$

Let us consider the one-meson part of $(\alpha\beta|\,H|\,\alpha'\beta')$. The expression

$$\begin{aligned}-\sum_{kk'q\mu}e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}}\,&\frac{\langle\alpha\,|\,V_{1k}^0\,|\,\mu q\rangle\langle\mu q\,|\,V_{1k'}^0\,|\,\alpha\rangle}{k_0k_0'(q_0+k_0)(q_0+k_0')(k_0+k_0')}(u_\beta(2),\,[V_{2k}V_{2k'}(k_0q_0+k_0'q_0+2k_0k_0')\\&+2V_{2k'}V_{2k}(k_0q_0+k_0'q_0+k_0^2+k_0'^2+k_0k_0')]\,u_\beta(2)).\end{aligned}\tag{37}$$

corresponds to the case $n_1=1$, $n_2=0$.

The quantity $\langle\mu q|\,V_{1k}|\,\alpha\rangle$ may be expressed through the phases δ_{ij} of the scattering of a meson on a nucleon.[2] Since according to experiment the phase δ_{33} is dominant over a broad energy region, we will neglect all phases except δ_{33}. Then,

$$\langle\mu q\,|\,V_k^0\,|\,\alpha\rangle\approx-\frac{2\pi}{(k_0q_0)^{1/2}q^3}\,\frac{v(k)}{v(q)}\,e^{i\delta_{33}(q)}\sin\delta_{33}(q)\,(u_\mu,\ P_{33}(qk)\,u_\alpha),\tag{38}$$

where P_{33} is the projection operator onto state 33: $P_{33}(qk)=\left(\delta_{kq}-{}^1\!/_3\,\tau_q\tau_k\right)(3(\mathbf{qk})-(\sigma\mathbf{q})(\sigma\mathbf{k}))$. After substituting (38) into (37) we see the cases $n_1=0$, $n_2=1$ and $n_1=1$, $n_2=0$ give the same results. If we neglect terms in e^{-3R} in the interaction energy, the one-meson part of $(\alpha\beta|\,H|\,\alpha'\beta')$ leads to the potential

$$\begin{aligned}W_2'=-\frac{4f^2}{(2\pi)^6}\int d^3k\,d^3k'\,dq\,&\frac{v^2(k)\,v^2(k')\sin^2\delta_{33}(q)\,e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}}}{(q_0+k_0)(q_0+k_0')(k_0+k_0')\,v^2(q)\,q^2q_0k_0^2k_0'^2}\Big\{\frac{1}{3}[3+(\tau_1\tau_2)]\,[2(\mathbf{kk}')^2(4q_0k_0+3k_0k_0'+2k_0'^2)\\&+(\sigma_1[\mathbf{kk}'])(\sigma_2[\mathbf{kk}'])\,k_0(k_0'-2k_0)]-[(\tau_1\tau_2)-1]\,[2(\mathbf{kk}')^2-(\sigma_1[\mathbf{kk}'])(\sigma_2[\mathbf{kk}'])(2k_0q_0+2k_0^2+k_0k_0')]\Big\}.\end{aligned}\tag{39}$$

Thus, the principal term (30) in the energy E_σ is connected with the potential $W_0 + W_1 + W_2'$.

We will now consider the last term in the energy E_σ [Eq. (16)]. In order to calculate it, we must find the matrix element

$$(\Phi^q_{\mu\nu},\ H\Phi_{\alpha'\beta'}), \tag{40}$$

where Φ^q is given by Eq. (A.6). The calculation of (40) is made easier by the fact that we are only interested in potentials of the order e^{-2R}. Therefore, we need retain in (40) only those terms decreasing no more rapidly than e^{-R}. We have

$$(H - q_0 - i\varepsilon)^{-1} V_{1q} F(1,\bar{a})\, F(2,\bar{a}) \approx F(2,\bar{a})\,(H_0 + U_1 - q_0 - i\varepsilon)^{-1} V_{1q} F(1,\bar{a}) + \ldots, \tag{41}$$

where the unwritten terms lead to potentials of higher order. For the same reason, the first term is sufficient in an expansion of the type (22). As a result of the calculations,

$$(\Phi^q_{\mu\nu},\ H\Phi_{\alpha'\beta'}) \approx (F_\mu(1,\bar{a}_1)\, F^q_\nu(2,\bar{a}_2) + F^q_\mu(1,\bar{a}_1)\, F_\nu(2,\bar{a}_2),\ [U_1^+(a_2) + U_2^+(a_1)]\, F_{\alpha'}(1,\bar{a}_1)\, F_{\beta'}(2,\bar{a}_2)), \tag{42}$$

where $F^q_\mu(1, \bar{a}_1)$ is the one-nucleon functional of a state with one meson. Using Eqs. (25), (26), (38), and (42), we find that the last term in (16) is

$$-\sum_{q\mu\nu} \frac{1}{q_0} \left| (\Phi^q_{\mu\nu},\ H\Phi^\sigma) \right|^2 = W''_{2\sigma}, \tag{43}$$

where $W''_{2\sigma}$ is the mean value of the potential W_2'':

$$\begin{aligned} W_2'' = -\frac{4}{3}\,\frac{f^2}{(2\pi)^6} \int d^3k\,d^3k'\,dq\,e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}}\, &\frac{(2k_0 + q_0)(2k_0' + q_0)\, v^2(k)\, v^2(k') \sin^2\delta_{33}(q)}{k_0^2 k_0'^2 q_0^2 q^2 v^2(q)\,(q_0 + k_0)(q_0 + k_0')} \\ \times [3 + (\tau_1\tau_2)]\,\{2(\mathbf{kk}')^2 + (\sigma_1[\mathbf{kk}'])(\sigma_2[\mathbf{kk}']) &- {}^7/_9\,(\sigma_1\mathbf{k}')(\sigma_2\mathbf{k})(\mathbf{kk}') \\ + {}^4/_9\,k^2(\sigma_1\mathbf{k}')(\sigma_2\mathbf{k}') + {}^1/_9\,k^2k'^2 &- {}^1/_9\,(\mathbf{kk}')^2(1 + (\sigma_1\sigma_2))\}. \end{aligned} \tag{44}$$

The total potential W is the sum of expressions (33), (36), (39), and (44):

$$W = W_0 + W_1 + W_2' + W_2''. \tag{45}$$

The following assumptions were made in the derivation of Eq. (45): (a) states with two or more (real) mesons were neglected, (b) it was implied that the various terms in the potential could be characterized by the way in which they decrease with distance (the potential of 2n-th order decreases as e^{-nR}), the computed potential W contains all terms of the second and fourth orders, (c) the phase δ_{33} is considered to be dominant and the remaining phases are neglected. It is not difficult, however, to take account of the remaining phases. Moreover, the two-nucleon potential was defined by us as a potential between nucleons with meson clouds, which led to the appearance of the denominator $1 + N_\sigma$ in the expression for the energy [Eqs. (16) and 30].

The basic difference between potential (45) and other known potentials[4-6] is contained in the terms W_2' and W_2'', which depend on the phase of the scattering of π mesons on nucleons. These terms have not been previously obtained. W_2'' differs from zero in the triplet charge state. The magnitude of W_2' is considerably greater in the triplet charge state than in the singlet state. It may therefore be expected that the terms W_2' and W_2'' will be imprtant in the triplet states.

The potential of Taketani et al.[4] represents the without-meson part of the term (30), i.e., $W_{0\sigma} + W_{1\sigma}/(1 + N_\sigma)$. In the transition to Eq. (36), we discarded the term N_σ in the denominator. The denominator in the potential of Taketani et al.[4] contains in addition the renormalized term $3\Delta_2'$, which should not be there according to the point of view of the present work.

The potential of fourth order V_4 of Brueckner and Watson[5] and of Gartenhaus[6] corresponds rather to the potential between nucleons without clouds. This potential may be obtained from the without-meson part of (30) by discarding N_σ in the denominator. In our notation $V_4 = W_1 + NW_0$.

In conclusion, I thank Academician V. A. Fock for valuable discussions.

APPENDIX

THE FUNCTIONALS Ψ^0 AND Ψ^q AS $R \to \infty$

The operator a_q is an operator of differentiation with respect to $\bar{a}_q$. Since $F(1, \bar{a})$ commutes with $\partial F(2, \bar{a})/\partial \bar{a}_q$, we have

$$H_0 F(1,\bar{a}) F(2,\bar{a}) = F(1,\bar{a}) H_0 F(2,\bar{a}) + F(2,\bar{a}) H_0 F(1,\bar{a}); \tag{A.1}$$

$$U_1^+ F(1,\bar{a}) F(2,\bar{a}) = F(2,\bar{a}) U_1^+ F(1,\bar{a}) + F(1,\bar{a}) U_1^+ F(2,\bar{a}), \tag{A.2}$$

where U_1^+ is the annihilating part of the operator U_1. Hence, in view of Eq. (7),

$$(H - 2E_0) F(1,\bar{a}) F(2,\bar{a}) = F(1,\bar{a}) U_1^+ F(2,\bar{a}) + F(2,\bar{a}) U_2^+ F(1,\bar{a}). \tag{A.3}$$

The dependence of the right hand side of (A.3) on R may be evaluated. Using Eq. (25), we have

$$U_2^+ F(1,\bar{a}) = -\sum_q V_{2q}^0 \frac{e^{-i\mathbf{q}\mathbf{R}}}{H_0 + U_1 + q_0 - E_0} V_{1q}^{0+} F(1,\bar{a}). \tag{A.4}$$

On the right hand side of Eq. (A.4), we have the integral

$$\int \frac{(\sigma_1 \mathbf{q})(\sigma_2 \mathbf{q})}{(\varepsilon_n + q_0) q_0} e^{-i\mathbf{q}\mathbf{R}} d^3 q, \quad \varepsilon_n > 0, \tag{A.5}$$

which decreases exponentially as $R \to \infty$. Hence, Eq. (12) follows. For large R, the one-meson state Ψ^q is

$$\Psi^q(1, 2, \bar{a}) \to \Phi^q(1, 2, \bar{a}) = \{a_q^+ - (H - q_0 - 2E_0 \pm i\varepsilon)^{-1} (V_{1q}^0 e^{i\mathbf{q}\mathbf{r}_1} + V_{2q}^0 e^{i\mathbf{q}\mathbf{r}_2})\} \Phi(1, 2, \bar{a}), \tag{A.6}$$

which can be verified also by using Eq. (A.3).

Note added in proof (September 16, 1957). After this work had been sent to the publisher, the author obtained Phys. Rev. 104, No. 6, 1956, with articles by Miyazawa and by Klein and McCormick, in which the two-nucleon problem is also considered. The approach to the problem and the method of our article is completely different from that developed in the articles of Miyazawa and of Klein and McCormick.

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A NEW POSSIBILITY IN NEUTRINO THEORY

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In addition to the usual description of electrons and other fermions by means of spinors of the second kind, it is suggested that for the neutrino, use can be made of Cartan spinors of the first kind with other transformation laws under reflection. Despite the conservation of parity, a number of conclusions of such a theory are analogous to the Yang-Lee theory.

In connection with the recent discovery of a number of phenomena[1] which are interpreted most naturally as a violation of the law of conservation of parity (decay of τ and θ mesons, $\pi - \mu$ mesons and the β-decay of oriented nuclei), it is necessary to investigate one more possible description of the neutrino, which may possibly permit the explanation of part of the abovementioned phenomena, and at the same time preserve all the conservation laws.

Let us assume that the transformation laws of the neutrino wave functions under reflection differ in a definite way from those of the usual spinors, such as electrons, for example. We will call spinors with special transformation properties pseudospinors. Cartan[2] pointed out the possibility of the existence in a world with an even number of dimensions, of pseudospinors, forming a special representation of the complete Lorentz group ("spinors of the first kind"*).

We will use the following notation:

$$ab = a_\mu b_\mu = a_i b_i - a_0 b_0; \quad \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = -2\delta_{\alpha\beta}(\delta_{00} = -1, \ \delta_{11} = \delta_{22} = \delta_{33} = 1).$$
$$\gamma_0 = \tilde{\gamma}_0 = \overset{*}{\gamma}_0 = \gamma_0^+$$

(the symbols next to the matrices represent successively the transpose, the complex conjugate and the Hermite conjugate);

$$\gamma_i^+ = -\gamma_i; \quad \overset{*}{\gamma}_2 = -\gamma_2,$$

all the remaining matrices are real;

$$\gamma_5 = -\gamma_5^+ = \gamma_0\gamma_1\gamma_2\gamma_3; \quad \gamma_5^2 = -1;$$

the charge conjugated spinor is $\psi^c = \gamma_2\psi^*$. Let the usual spinors corresponding to the electron-positron field (Cartan's spinors of the second kind) transform under reflection with respect to a hyperplane perpendicular to the unit vector a_μ ($a_\mu^2 = \pm 1$), as

$$\psi \rightarrow \psi' = \pm L(a_\mu)\psi, \quad L(a_\mu) = a_\mu\gamma_5\gamma_\mu; \quad L^2 = -a_\mu^2.$$

This choice is unique to within i with the requirement that the relation connecting ψ and ψ^c be invariant.[3] Here, pseudospinors will be transformed by the matrix

$$L'(a_\mu) = a_\mu\gamma_\mu.$$

The properties of the usual spinors and the pseudospinors are identical under rotations. If a vector and pseudovector are constructed of pseudospinors (with respect to spatial reflections† the structure will be the same as in the usual case), it is easy to verify that the matrix γ_5 is added to the scalar, pseudoscalar and tensor constructed of pseudospinors. In particular, the Dirac equation for a free pseudospinor field is replaced by the modified equation

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*The properties of pseudospinors in connection with β-decay were investigated in detail by Brodskii at one of the lectures at the Moscow State University (1951).

†We recall that with regard to reflections with a change in the sign of time, the known invariant combinations behave as reciprocal pseudoquantities, for example $\psi^+\gamma_0\psi$ as a pseudoscalar and $\psi^+\gamma_0\gamma_5\psi$ as a scalar, because of the condition $-1 = a_\mu^2$ for a timelike a_μ.

$$(-i\gamma_\mu \partial / \partial x^\mu + i\gamma_5 (im_0))\psi = 0, \tag{1}$$

where the factor i is placed in front of the real m to conserve the correct relation between the energy-momentum and the mass (when the Hermitian nature of the Lagrangian is violated). For m = 0, Eq. (1) is outwardly indistinguishable from the corresponding Dirac equation. Now let the neutrino field ψ_ν be a pseudospinor. The β-decay interaction is described, as is well known, in the form of products of a scalar, vector, etc., constructed from nucleon functions multiplying respectively a scalar, vector, etc., constructed from the functions ψ_e and ψ_ν.

Let us define the latter quantities in our case, limiting ourselves for the time being to invariance under spatial reflections ($a_\mu^2 = 1$). As follows from elementary calculations, the mixed scalar will be $\psi_e^+\gamma_0(1+\gamma_5)\psi_\nu$, the pseudoscalar $\psi_e^+\gamma_0(1-\gamma_5)\psi_\nu$, the vector $\psi_e^+\gamma_0\gamma_\mu(1+\gamma_5)\psi_\nu$, the pseudovector $\psi_e^+\gamma_0\gamma_\mu(1-\gamma_5)\psi_\nu$, and the tensor $\psi_e^+\gamma_0\gamma_\mu\gamma_\nu(1+\gamma_5)\psi_\nu$. Indeed, for example,

$$(L\psi_e)^+\gamma_0(1+\gamma_5)(L'\psi_\nu) = a_\mu a_\nu \psi_e^+ \gamma_\nu^+ \gamma_5^+ \gamma_0 (1+\gamma_5)\gamma_\mu\psi_\nu = -a_\mu a_\nu \psi_e^+ \gamma_0 (1+\gamma_5)\gamma_\mu\gamma_\nu\psi_\nu = a_\mu^2 \psi_e^+ \gamma_0 (1+\gamma_5)\psi_\nu = \psi_e^+\gamma_0(1+\gamma_5)\psi_\nu.$$

In all of these expressions, ψ_e and ψ_ν may be replaced simultaneously by anti-particle (charge conjugated) quantities.*

Let us now investigate the invariance with respect to time inversion and anti-particle (charge) conjugation. Just as in the case of the usual invariants, invariants composed of pseudospinors, or spinors and pseudospinors, will behave as reciprocal pseudoquantities under time inversion (i.e., a scalar will become a pseudoscalar, etc.). In distinction from other types of interaction, this circumstance does not impose any kind of compulsory supplementary conditions on the coefficients of the various terms in the β-decay interaction Lagrangian L_{int}, because all the quantities in L_{int} composed of ψ_ν and ψ_e enter in the form of a product multiplying the corresponding nucleon combination, which also change their character in the transition from space to time inversions. As regards symmetry with respect to anti-particle conjugation, in all the usual interactions, with the exception of β-decay, it is a necessary consequence of the invariance with respect to time and space reflections and therefore need not necessarily be considered a compulsory condition (see the Luders-Pauli theorem). If, nevertheless, conservation of symmetry with respect to anti-particle (charge) conjugation is demanded, then in the scalar case, for example, we must take the half-sum†

$$^1/_2\,(\psi_N^+\gamma_0\psi_N\cdot\psi_e^+\gamma_0(1+\gamma_5)\psi_\nu + \psi_N^{c+}\gamma_0\psi_N^c\cdot\psi_{\bar{e}}^{+}\gamma_0(1+\gamma_5)\psi_{\bar{\nu}}). \tag{2}$$

The investigation may be carried further as in the work of Lee and Yang.[1] However, since in the present approach we assume conservation of parity, the coefficient of asymmetry in β-decay will be determined only by the Coulomb correction, proportional to Ze^2. This also results in the present case from the relations between the coefficients $C_S' = iC_S$, $C_P' = -iC_P$, etc. If experiment confirms the presence in the β-decay of oriented nuclei of an asymmetry due to nonconservation of parity, i.e., to a non-Coulomb term, the possibility suggested herein of using pseudospinors must be rejected, but it will be desirable to base this circumstance on general arguments.

On the other hand, the asymmetry observed by Lederman and co-workers in the successive decay of pions and muons with the emission of a neutrino, is also preserved with the use of pseudospinors.

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*A special situation arises if we choose $L^2 = +1$ instead of $L^2 = -1$ as is done above. In this case, as is well known, the antineutrino may be identified with the neutrino $\psi_{\nu} = \psi_{\gamma}^{c} = \gamma_2\gamma_{\nu}^{*}$ (Majorana variant) with corresponding consequences for the invariants.

†In the case in which L_{int} is symmetrized only with respect to electron-positrons and nucleon-antinucleons, instead of Eq. (2) we obtain

$$^1/_2\,(\psi_N^+\gamma_0\psi_N\cdot\psi_e^+\gamma_0(1+\gamma_5)\psi_\nu + \psi_N^{c+}\gamma_0\psi_N^c\psi_{\bar{e}}^{+}\gamma_0(1-\gamma_5)\psi_\nu),$$

which is outwardly equivalent to invariance with respect to the simultaneous transformations of space inversion and charge conjugation, with violation of each of them separately, if the special character of the transformations of the pseudospinors ψ_ν is not taken into account.

We also note that the pseudospinor treatment of the neutrino leads to the Salam condition[5] of invariance with respect to the transformation $\psi_\nu' \rightarrow \gamma_5\psi_\nu$, introduced by him as a postulate. Also, the possibility of applying pseudospinors to other fermions is not excluded.

Note added in proof (Sept. 18, 1957). We should mention the interesting possibility of mixed spinors of the first kind with respect to space (time) reflections and of the second kind with respect to time (space) reflections. We also note that in the case of nonconservation of parity with invariance with respect to the Salam transformation,[5] a new law of conservation of "neutrino charge" holds, with a current density defined by a pseudovector.

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NONLINEAR THEORY OF PHASE OSCILLATIONS INDUCED IN ELECTRON SYNCHROTRONS BY QUANTUM RADIATION FLUCTUATIONS

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A nonlinear theory of phase oscillations induced by radiation fluctuations in electron synchrotrons is considered. It is shown that the nonlinear theory gives essential correction to results of the linear theory. The nonlinear theory predicts greater electron losses and imposes upon synchrotron parameters stronger restrictions than does a linear theory.

It is well known[1] that the quantum nature of radiation causes phase oscillations of electrons in synchrotrons. The mechanism of exciting these oscillations is similar to that described by Sokolov and Ternov[2,3] for exciting betatron oscillations.

Sands[1] calculated the mean square deviation of the electron phase oscillations induced by radiation for a weak-focusing synchrotron. The calculation was performed in the linear approximation with a neglect of the existence of limits to the phase oscillations. Such a calculation is valid only for small oscillations. These results have been generalized in the same approximation to the case of strong-focusing synchrotrons.[4-6]

The linear theory of phase vibrations treated by Sands,[1] Kolomenskii,[4] and the present author[5,6] is valid only for small deviations of the electron phase from its equilibrium value. In general, however, this assumption does not resolve the fundamental problem, that of determining the loss of electrons caused by these oscillations. In order that an electron be removed from further acceleration, it is necessary that its phase of oscillation be outside the allowable limits. As is well known, the left limit is φ_s, the negative of the equilibrium phase. The boundary φ_1 on the right is given by

$$\sin\varphi_1 + \sin\varphi_s - (\varphi_1 + \varphi_s)\cos\varphi_s = 0. \tag{1}$$

Thus in order that an electron reach the limits, its phase must change by $2\varphi_s$ to the left or by $2\varphi_1 - \varphi_s$ to the right, and these are relatively large durations. If, for instance, $\varphi_s = 45°$, these deviations must be $2\varphi_s \approx \pi/2$ and $\varphi_1 - \varphi_s \approx \pi/4$, respectively, which can in no sense be considered small. The linear approximation is therefore insufficient for an investigation of the theoretically fundamental problem of loss of electrons due to phase changes. One may thus question the accuracy of the predictions of the linear theory, and it becomes necessary to formulate a nonlinear theory without assuming small deviations of the electron phase from the equilibrium value.

In the present article we shall consider certain problems of the nonlinear theory of phase oscillations induced by quantum radiation fluctuations.

The method for deriving the nonlinear stochastic equation describing phase oscillations induced by radiation is entirely analagous to that for obtaining the linear stochastic expression, Eq. (2) of one of the author's previous works.[6] This nonlinear stochastic equation is

$$\ddot{\psi} + \gamma\dot{\psi} + f^2 [\cos\varphi_s - \cos(\varphi_s + \psi)] = \frac{k\omega\alpha}{\lambda E_s}\Big[W_s - \sum_i \varepsilon_i \delta(t - t_i)\Big], \tag{2}$$

where $\psi = \varphi - \varphi_s$, φ is the phase with which the electron passes through the high-frequency field, φ_s is the equilibrium value of this phase, ϵ_1 is the energy of the photon emitted at the instant i, k is the number of the accelerating harmonic of the high-frequency field, R_0 is the radius of curvature of the curved sections of the synchrotron, $\lambda = 1 + L/2\pi R_0$, L is the total length of the linear sections of the synchrotron per revolution,

$$\omega = c/R_0\lambda,\ f^2 = (k\omega^2\alpha/2\pi\lambda)(ev_0/E_s),\quad \gamma = (4 - \alpha)(2\omega r_0/3R_0)(E_s/mc^2)^3,$$
$$r_0 = e^2/mc^2,\ \delta\langle R\rangle/\langle R\rangle = \alpha\delta E/E,\ W_s = (2ce^2/3R_0^2)(E_s/mc^2)^4,$$

and ev_0 is the amplitude of the high-frequency field.

If ψ is assumed small, Eq. (2) can be linearized and leads to Eq. (2) of the work cited.[6]

As is well known, the problem is generally solved in the following way. One first obtains a solution to the homogeneous Eq. (2) in the form of a Fourier series whose coefficients and frequencies depend on some parameter. The right side of the equation is then treated as a small perturbation, and the effect of this perturbation on the small parameter of the homogeneous solutions is investigated to find its effect on the solutions obtained. Accounting for the chance nature of the small perturbation makes it possible to clarify the statistical properties of the desired solutions.

This method of solution is hardly practical, however, in view of the many calculations necessary, even with computers. We shall therefore use a somewhat different method which is sufficient to answer the most important questions.

We first change to a new independent variable ξ and to a new function z in Eq. (2), using

$$\xi = \int^t f\,dt,\ \psi = uz,\ u = \exp\Big(-\frac{1}{2}\int^{\xi} q\,d\xi\Big),\ q = \frac{f'}{f} + \frac{\gamma}{f}, \tag{3}$$

where the primes indicate differentiation with respect to ξ. Then Eq. (2) becomes

$$z'' + \frac{1}{u}[\cos\varphi_s - \cos(\varphi_s + uz)] - \Big(\frac{1}{4}q^2 + \frac{1}{2}q'\Big)z = \frac{k\omega\alpha}{\lambda E_s uf}[W_{1s} - \sum_i \varepsilon_i \delta(\xi - \xi_i)], \tag{4}$$

where

$$W_{1s} = W_s/f.$$

In this equation the term

$$-(q^2/4 + q'/2)z \tag{5}$$

can always be neglected. Indeed, denoting $\Delta_c E_s$ the synchronous energy change per period of synchrotron oscillation, we obviously obtain

$$\frac{f'}{f} \sim \frac{\Delta_c E_s}{E_s} \ll 1,\ \frac{\gamma}{f} \sim \Big[\frac{r_0}{R_0}\Big(\frac{E}{mc^2}\Big)^3\Big]^{1/2} \ll 1. \tag{6}$$

Therefore $|q| \ll 1$. Similarly, $|q'| \ll 1$.

The homogeneous equation corresponding to (4) can be written

$$z'' + u^{-1}[\cos\varphi_s - \cos(\varphi_s + uz)] = 0. \tag{7}$$

We note further that $u' = -qu/2 \ll u$. We therefore multiply (7) by z' and integrate the product, bearing in mind the above inequality, thus obtaining a constant of the motion in the form

$$Q = {}^1\!/_2\, z'^2 + u^{-2}[uz\cos\varphi_s - \sin(\varphi_s + uz)]. \tag{8}$$

It follows from (8) that bound states will exist only when the constant of the motion satisfies

$$-u^{-2}\sin\varphi_s = Q_{\min} \leqslant Q \leqslant Q_{\max} = u^{-2}(-2\varphi_s\cos\varphi_s + \sin\varphi_s). \tag{9}$$

If, however, Q for any reason leaves the interval given in (9), the particle moves to infinity; in other words it leaves the accelerating process and is lost.

Let us now return to Eq. (4). It is seen from this equation that the mechanism of exciting phase oscillations consists of discretely changing, by emission of photons, the rate at which the phase of a particle varies. It is also seen that emission of a photon of energy ϵ produces a change $\Delta z'$ of magnitude

$$\Delta z' = -(k\omega\alpha/\lambda E_s u f)\,\varepsilon. \tag{10}$$

Equation (8), on the other hand, shows that the mean change that chance discontinuities $\Delta z'$ cause in the constant of motion is given by

$$\langle\Delta Q\rangle = \langle z'\Delta z'\rangle + {}^1\!/_2\langle(\Delta z')^2\rangle = {}^1\!/_2\langle(\Delta z')^2\rangle, \tag{11}$$

which can be rewritten, using (10):

$$\langle\Delta Q\rangle = (k^2\omega^2\alpha^2/2\lambda^2 E_s^2 u^2 f^2)\langle\varepsilon^2\rangle. \tag{12}$$

It follows that if at $\xi = 0$ the electron is at rest on the bottom of the potential well given by (9), the mean value of Q for other values of ξ will be

$$\langle Q\rangle = -\frac{1}{u^2}\sin\varphi_s + \frac{k^2\omega^2\alpha^2}{2\lambda^2}\int_0^{\xi}\frac{\langle\varepsilon^2\rangle}{E_s^2u^2f^2}\,d\xi. \tag{13}$$

In order that radiation-induced phase oscillations cause only small losses, we must have $\langle Q\rangle \ll Q_{\max}$. According to Eq. (13) this can be written

$$\frac{k^2\omega^2\alpha^2u^2}{4\lambda^2}\int_0^{\xi}\frac{\langle\varepsilon^2\rangle}{E_s^2f^2u^2}\,d\xi \ll \sin\varphi_s - \varphi_s\cos\varphi_s. \tag{14}$$

The stronger the inequality (14), the lower the radiation-induced electron losses. It is clear that for $\langle Q\rangle = Q_{\max}$ practically all the electrons will be lost.

Performing the calculations indicated in (14), we have

$$\frac{55\pi}{48\sqrt{3}}\,\frac{k\alpha\omega}{\lambda}\left(\frac{r_0}{R_0}\right)^2\frac{hc}{e^2}\sqrt{\frac{E_s}{ev_0}}\int_0^t \exp\left(-\int_\tau^t\gamma\,d\tau\right)\left(\frac{E_s}{mc^2}\right)^5\left(\frac{E_s}{ev_0}\right)^{1/2}d\tau \ll \sin\varphi_s - \varphi_s\cos\varphi_s. \tag{15}$$

If we now note that $1/\gamma$ is much greater than the period of oscillations of the system we are treating and is much larger than the period of acceleration, we can show that

$$\int_0^t \exp\left(-\int_\tau^t\gamma\,dt\right)\left(\frac{E_s}{mc^2}\right)^5\left(\frac{E_s}{ev_0}\right)^{1/2}d\tau \ll \frac{3R_0}{2(4-\alpha)\,r_0\omega}\sqrt{\frac{E_s}{ev_0}}\left(\frac{E_s}{mc^2}\right)^2, \tag{16}$$

in which equality is achieved only if (a) the left side increases monotonically, in which case the equality occurs only asymptotically at infinity, or (b) if the left side has a maximum, in which case the equality is attained at this maximum. We note that in actual accelerators it is case (a) which is realized. The proof of (16) and the corrolary assertions follow immediately from the identity

$$\frac{d}{dx}\int_0^x \exp\left(-\int_\tau^x\gamma\,d\tau\right)f(\tau)\,d\tau = f(x) - \gamma\int_0^x \exp\left(-\int_\tau^x\gamma\,d\tau\right)f(\tau)\,d\tau. \tag{17}$$

On the basis of (16), Eq. (15) can be written in the convenient form

$$F(E) \ll f_1(\varphi_s), \tag{18}$$

where

$$F(E) = \frac{55\pi}{16\sqrt{3}} \frac{k\alpha}{\lambda(4-\alpha)} \frac{r_0}{R_0} \frac{hc}{e^2} \frac{mc^2}{ev_0} \left(\frac{E_s}{mc^2}\right)^3, \tag{19}$$

$$f_1(\varphi_s) = 2(\sin\varphi_s - \varphi_s \cos\varphi_s). \tag{20}$$

This is the fundamental requirement which must be fulfilled according to the nonlinear theory if the electron losses due to phase oscillations induced by quantum radiation fluctuations are to be small.

Let us compare the linear and nonlinear criteria. According to Eq. (8) of the previously cited work,[6] the criterion of the linear theory can be written

$$F(E) \ll (l^2/8)\sin\varphi_s = f_2(\varphi_s), \tag{21}$$

where ℓ denotes the region of permissible phase oscillations. For various values of φ_S, f_1 and f_2 are

φ_s	15°	30°	45°	60°
$f_1(\varphi_s)$	0.011	0.09	0.3	0.7
$f_2(\varphi_s)$	0.018	0.15	0.5	1.2

One thus obtain, approximately,

$$f_2/f_1 \approx 1.7. \tag{22}$$

This means that (18), the requirement given by the nonlinear theory, is stronger than (21), that of the linear theory.

As an example let us consider an unsegmented weak-focusing synchrotron whose parameters are $n = 0.6$, $R_0 = 400$ cm, $ev_0 = 100$ kev, and $k = 4$. If we then express E in Bev, we have $F(E) = 0.16\ E^3$. To be specific let us take the equilibrium phase to be $\varphi_S = 45°$. One then sees that for the linear theory the left side of (21) will be about half the right side at 1.3 Bev. The nonlinear theory, on the other hand, makes the left side of (18) about half the right side at 1 Bev. The linear theory thus leads to the conclusion that under the above conditions the synchrotron will cease operating somewhere in the neighborhood of 1.3 Bev. The nonlinear theory, however, leads to the conclusion that the synchrotron will actually stop working at an energy near 1 Bev.

The nonlinear theory thus gives essential corrections to the linear theory. We note that the nonlinear theory of phase oscillations induced by quantum radiation fluctuations verifies the conclusions[5-7] that strong focusing is unavoidable at electron energies of several Bev.

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THE RELATION BETWEEN MATRIX ELEMENTS FOR "PARTICLES" AND "HOLES" IN NUCLEAR SHELL THEORY

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The relation between matrix elements for "particles" and "hole" configurations is considered for the case of two-particle operators which appear in a general non-central interaction. The particular cases of central, tensor, and spin-orbit forces are discussed. The relation between matrix elements of one-particle operators is briefly discussed.

A configuration of the type j^{m-n}, with $n < m/2$ (where m is the total number of states in the j shell) is called a "hole" configuration in the j-shell. The properties of states of the configurations j^n and j^{m-n} are similar in many ways. The relation between the matrix elements for the configurations ℓ^n and ℓ^{m-n} where m is the total number of states in the ℓ-shell, was investigated by Racah[1] for atoms, for one-particle operators and for the special case of two-particle operators in which we have central pair interactions. In the present note, with this same end in view, we analyze the two-particle and one-particle interactions which occur in nuclear theory, for the cases of j–j and L–S coupling.

1. TWO-PARTICLE OPERATORS

The operator for any central or non-central interaction can be represented as [1-3]

$$V(1,2) = \{S^k(1,2)\cdot L^k(1,2)\} = \sum_{uv}\{S^k(s^{u_1}(1)\cdot s^{u_2}(2))\cdot L^k(l^{v_1}(1)\cdot l^{v_2}(2))\}, \tag{1}$$

where $\{A^k \cdot B^k\}$ is the scalar product of two tensor operators,[1] $S^k(s^{u_1}(1)\cdot s^{u_2}(2))$ is the tensor product[2,3] of rank k of two single-particle tensor operators of rank u_1 and u_2, acting in the spin spaces of particles 1 and 2, respectively, and $L^k(\ell^{v_1}(1)\cdot \ell^{v_2}(2))$ is a similar product of orbital operators:

$$L_q^k = \sum_p C^{v_1 v_2 k}_{p,q-p,q}\, l_p^{v_1}(1)\, l_{q-p}^{v_2}(2).$$

For central interactions, $k = 0$, while for non-central interactions, $k \neq 0$. For example, for tensor forces[2-5] $k = 2$, for spin-orbit forces[2,3,6,7] $k = 1$.

(a) Applying formulas of the Racah tensor algebra,[1,2,8,9] and using (1), we can always write the interaction operator in the form

$$\sum_{ij} V(ij) = \sum_{ij}\sum_r A_r\{t^r(i)\cdot t^r(j)\},$$

where r is the rank of the tensor operator in spin-orbit space. We know[1] that the expression for central pair interactions in L–S coupling has this same form. Therefore, there is a simple relation between the matrix elements for any central or non-central forces in j-j coupling for the configurations j^n and j^{m-n}, which is of exactly the same type as the relation between the matrix elements for central interactions for the configurations ℓ^n and ℓ^{m-n} (Cf. Refs. 1, 10, 11, and below). In particular, if j–j coupling is valid and there are only pair interactions, then we should get the same level schemes for Li^6 and N^{14}, Ca^{43} and Ca^{45} etc. (The last example is discussed briefly[12] in connection with the analysis of the structure of nuclei of the 1 $f_{7/2}$ shell.)

(b) The case of L–S coupling is much more complicated. We rewrite expression (1) in the form

$$\sum_{ij} V(ij) = \sum_{ij}\sum_{uv}\{S^k(s^{u_1}(i), s^{u_2}(j))\cdot L^k(l^{v_1}(i)\cdot l^{v_2}(j))\} = \sum_{ij}\sum_{uv} V_0^{kk}(t^{u_1v_1}(i)\cdot t^{u_2v_2}(j)) = \frac{1}{2}\sum_{uv} V_0^{kk}(T^{u_1v_1}\cdot T^{u_2v_2})$$
$$-\frac{1}{2}\sum_{uv}\sum_{i} V_0^{kk}(t^{u_1v_1}(i)\cdot t^{u_2v_2}(i)). \tag{2}$$

for the case of n particles.

Here t^{uv} is a double tensor of rank u in spin space and rank v in orbital space, $V_0^{kk}(t^{u_1v_1}\cdot t^{u_2v_2})$ is the scalar part of the double tensor product $V^{kk}(t^{u_1v_1}\cdot t^{u_2v_2})$ of rank k in spin space and rank k in orbital space; $T^{uv} = \sum_i t^{uv}(i)$; the terminology is the same as in Racah's paper.[1] The purpose of the transformation (2) is to express the two-particle operator in terms of one-particle operators so that we can later make use of the relations between matrix elements of one-particle operators for the ℓ^n and ℓ^{m-n} configurations which were derived by Racah[1] and which are also discussed below. Using tensor algebra and fractional parentage coefficients[3,13,14] for calculating the matrix elements, we arrive after very involved calculations at the result:

$$\langle l^n\alpha LSJM|\,2\sum_{ij}V(i,j)\,|l^n\alpha' L'S'JM\rangle = (-1)^{L+S'-J}\,W(LSL'S';Jk)\,[k]\sum_{uv}(-1)^{u_1+v_1+u_2+v_2}\sum_{\alpha'' L'' S''} W(v_1v_2LL';kL'')$$
$$W(u_1u_2SS';kS'')\,\langle l^n\alpha LS\|T^{u_1v_1}\|l^n\alpha''L''S''\rangle\langle l^n\alpha''L''S''\|T^{u_2v_2}\|l^n\alpha'L'S'\rangle - \langle l^n\alpha LSJM\Big|\sum_i X_0^{kk}(i)\Big|\,l^n\alpha'L''S''JM\rangle \tag{3}$$
$$\equiv\sum_{uv}A(u,v)-\sum_{uv}B(u,v).$$

X^{kk} is defined by the matrix

$$\langle l\,{}^1/_2\|X^{kk}\|l\,{}^1/_2\rangle = [k]\sum_{uv}(-1)^{u_1+v_1+u_2+v_2}\,W(v_1v_2ll;kl)\;W(u_1u_2\,{}^1/_2\,{}^1/_2;k\,{}^1/_2)\,\langle l\,{}^1/_2\|t^{u_1v_1}\|l\,{}^1/_2\rangle\langle l\,{}^1/_2\|t^{u_2v_2}\|l\,{}^1/_2\rangle,$$

$[k] = 2k + 1$, and the other symbols are the same as Racah's.[1] To proceed further, we fix u_1, v_1, u_2, and v_2 on the right hand side of (3). When we go over from the ℓ^n configuration to ℓ^{m-n}, we see that

$$A(u,v)\to(-1)^{u_1+v_1+u_2+v_2}A(u,v),\quad B(u,v)\to -B(u,v).$$

We now present some specific cases. k = 2 for tensor forces, and the last term on the right of (3), i.e., $B(u, v)$, becomes zero. It is easily seen that the tensor interaction operator contains only those one-particle operators t^{uv} $(u = 1)$, which change the parity of the particle state only if v is odd (for v = 1, it is a vector operator). Since all the matrix elements which appear in the expression $A(u, v)$ are diagonal with respect to configuration, v_1 and v_2 are even. Thus the following assertion holds for tensor forces: the matrix elements

$$\langle l^n\alpha LSJM\Big|\sum_{ji}V(ij)\Big|\,l^n\alpha'L'S'JM\rangle \;\text{и}\; \langle l^{m-n}\alpha LSJM\Big|\sum_{ij}V(ij)\Big|\,l^{m-n}\alpha'L'S'JM\rangle$$

are the same (α may be, for example, the seniority quantum number, or the isotopic spin T, or both; the matrix elements are diagonal in T). For central forces the expressions $A(u, v)$ also coincide for the ℓ^n and ℓ^{m-n} configurations; the expressions $B(u, v)$ are easily seen to be independent of L, S, L′, S′: $\sum_{u,v} B(u,v) = nB_0$ for the ℓ^n configuration, $\sum_{uv} B(u,v) = (m-n)B_0$ for the ℓ^{m-n} configuration.

So for central forces the "level schemes" are the same for the ℓ^n and ℓ^{m-n} configurations, though the absolute values of the "binding energies" differ by $(m - 2n)B$. As mentioned earlier, this fact is well known.[1,10,11]

Our last example is the spin-orbit force. It is not difficult to show that for any interactions with k = 1, two types of operators t^{uv} occur in the expansion (2): (a) for odd v the parity of the particle state changes (for v = 1, the operator is a vector), (b) for odd v the parity of the state is unchanged (for v = 1, the operator is a pseudovector).

Thus even though the transformation properties for particular pairs $t^{u_1v_1}$, $t^{u_2v_2}$ are simple when we go from ℓ^n to ℓ^{m-n}, there is no overall simple relation. Both the tensor and spin-orbit forces give rise to spin-orbit splitting, but they behave completely differently; even if the spin-orbit forces are, roughly speaking, equivalent to a spin-orbit coupling of the type $\xi\sum_i \boldsymbol{\ell}_i\cdot \mathbf{S}_i$, with a constant ξ which increases with filling of the shell,[6,15] the tensor forces have no such clear connection with the Mayer-Jensen shell theory. All the theoretically admissible types of interaction have been enumerated in Refs. 16 and 17.

The relation between matrix elements for "particles" and "holes" can also be investigated by the same general methods for more complicated cases such as configuration mixing. The simplest example is the matrix element

$$\langle j^{m-1}\,T_1 J_1 \alpha_1,\, j_1,\, TJ\alpha \Big| \sum_{ij} V(ij) \Big| j^{m-1} T_1 J_1 \alpha_1, j_2,\, TJ\alpha' \rangle,$$

where $J_1 = j$, $T_1 = 1/2$, and α is the set of quantum numbers which are needed in addition to J and T for unique characterization of a state. This matrix element has previously been calculated[17,18] by other methods. The basis for the calculation is a formula which is related to formula (33) of Racah's paper[13] and derived in similar fashion:

$$\begin{aligned}\langle j^{n-1} T_1 J_1 \alpha_1,\, j_1,\, TJ\alpha \Big| \sum_{ij} V(ij) \Big| j^{n-1} T_1 J_1 \alpha_1,\, j_2,\, TJ\alpha' \rangle &= \delta(j_1, j_2) \langle j^{n-1} T_1 J_1 \alpha_1 \Big| \sum_{ij}{}' V(ij) \Big| j^{n-1} J_1 T_1 \alpha_1 \rangle \\ &+ (n-1) \sum_{T_2 J_2 \alpha_2 J_3 T_3} C^{j^{n-1} T_1 J_1 \alpha_1}_{j^{n-2} T_2 J_2 \alpha_2} C^{j^{n-1} T_1 J_1 \alpha_1}_{j^{n-2} T_2 J_2 \alpha_2} U(J_2 j J j_1; j J_3)\, U(J_2 j J j_2; j J_3) \qquad (4) \\ \times \{U(T_2\, {}^1\!/_2\, T\, {}^1\!/_2;\ {}^1\!/_2\, T_3)\}^2 \langle j j_1 J_3 T_3 | V(n-1, n) | j j_2 J_3 T_3 \rangle,&\quad U(abcd; ef) = \{[e]\,[f]\}^{1/2}\, W(abcd; ef).\end{aligned}$$

If $n = m$, from formula (19) of Racah's paper[13]

$$C^{j^{m-1},\, J_1 = j,\, T_1 = 1/2}_{j^{m-2} J_2 T_2} = \left\{ \frac{2}{m-1}\ \frac{[J_2]\,[T_2]}{[j]\,[T_1]} \right\}^{1/2}.$$

By substituting these values in (4), we obtain the simple final formula[18] after some transformations.

2. ONE-PARTICLE OPERATORS

Racah's[1] results require only slight modification (the inclusion of isotopic spin):

$$\begin{aligned}&\langle l^{m-n} \alpha LSTM_L M_S M_T \,|\, F^{uvr}_{\beta\gamma\delta} \,|\, l^{m-n} \alpha' L' S' T' M'_L M'_S M'_T \rangle \\ &= -(-1)^{u+v+T'-T} \langle l^n \alpha LSTM_L M_S - M_T \,|\, F^{uvr}_{\beta\gamma-\delta} \,|\, l^n \alpha' L' S' T' M'_L M'_S - M'_T \rangle.\end{aligned}$$

$F^{uvr}_{\beta\gamma\delta}$ is a triple tensor, r its rank in isotopic spin space, and β, γ, and δ are the indices of the components of the tensor. For operators which are diagonal in M_T (electromagnetic transitions, spin-orbit coupling operator $\xi \sum_i \ell_i \cdot S_i$ etc.), $\delta = 0$. For the operators of β-decay theory, $\delta \neq 0$. The state T, M_T of the configuration ℓ^n is related to the state T, $-M_T$ of the configuration ℓ^{m-n}, because, for example, a proton corresponds to a "proton hole" in the filled shell, i.e., to a nucleus with an odd neutron so the sign of M_T changes. Thus the magnetic moments of nuclei with configurations ℓ^n and ℓ^{m-n} are the same, while their quadrupole moments have opposite signs, as is well known.

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THE ENERGY DEPENDENCE OF A SCATTERING CROSS SECTION NEAR THE THRESHOLD OF A REACTION

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The energy dependence of the cross section for an elastic scattering process X(aa)X is examined at energies near the threshold for a reaction X(ab)Y (b being a neutron). It is shown that the scattering cross section has a singularity at the threshold. From the singularity, one can obtain the spin and parity of the nucleus Y formed in the reaction, and also simplify the phase analysis of the elastic scattering.

1. INTRODUCTION

In a recently published work[1] it was shown that the cross section for the elastic scattering of protons from Li^7, $\mathrm{Li}^7(\mathrm{pp})\mathrm{Li}^7$, has a peak at an energy corresponding to the threshold of the reaction $\mathrm{Li}^7(\mathrm{pn})\mathrm{Be}^7$. The peak is about 40 kev wide and at some angles its height is 20 — 30% of the cross section at the same angles. The existence of such a noticeable anomaly in an elastic scattering cross section makes it worth while to consider in general the behavior of the cross section for the elastic scattering X(aa)X at energies near the threshold E_{thr} of the reaction X(ab)Y. Taking advantage of the fact that the energy dependence of the reaction cross section near its threshold is known, one can use the unitarity of the scattering matrix to determine the energy dependence of the phases for scattering near threshold.

It turns out that the behavior of the elastic scattering near threshold gives information not only on the scattering itself, but also on the reaction. In particular, such experiments can be used to find the spin and parity of the particles formed in the reaction, and also greatly simplify the phase analysis of the elastic scattering near threshold. We consider the simplest case first, that all particles a, X, b, Y have spin zero.

2. SPINLESS PARTICLES

We write down the wave function at energies above threshold $(E \geq E_{\mathrm{thr}})$. At such energies, both elastic scattering X(aa)X and the reaction X(ab)Y are possible and the wave function has the asymptotic form

$$\Phi_a\Phi_X\left[e^{i\mathbf{k}_1\mathbf{r}} + \frac{1}{r}e^{ik_1r}\sum_l \frac{2l+1}{2ik_1}(S_l-1)P_l(\cos\theta)\right] - \Phi_b\Phi_Y\frac{e^{ikr}}{r}\sum_l \frac{2l+1}{2\sqrt{k_1k}}M_lP_l(\cos\theta), \qquad (2.1)$$

where Φ_a, Φ_b, Φ_X and Φ_Y are the internal wave functions of the particles a, b, X and Y; k_1 and k are the wave vectors describing the relative motion of a + X and b + Y, and the P_ℓ are Legendre polynomials. The first term in (2.1) describes the elastic scattering X(aa)X, while the second one refers to the reaction X(ab)Y. The matrix elements for scattering, S_ℓ, and those for the reaction, M_ℓ, satisfy the relation

$$|S_l|^2+|M_l|^2=1, \tag{2.2}$$

which expresses the conservation of particles. If there is no Coulomb interaction between b and Y, then it is known[2] that as $E \to E_{thr}$, M_ℓ behaves as follows:

$$M_l \to (kR)^{l+1/2}\mathfrak{M}_l, \quad k=(1/h)\sqrt{2\mu(E-E_{\text{п}})};\ \mu=m_bm_Y/(m_b+m_Y), \tag{2.3}$$

where R is the reaction radius, and $\mathfrak{M}_\ell$ is a constant, independent of energy. Substituting (2.3) in (2.2), we find that for small kR,

$$|S_l|=1-\frac{1}{2}(kR)^{2l+1}|\mathfrak{M}_l|^2\equiv 1-(kR)^{2l+1}a_l,$$

i.e.

$$S_l=S_l^{(0)}(1-(kR)^{2l+1}a_l), \tag{2.4}$$

where $|S_\ell^{(0)}| = 1$ for $E \geq E_{thr}$. The matrix elements S_ℓ are analytic functions of k, and hence formula (2.4) holds also for $E \leq E_{thr}$, at which energies k becomes imaginary, $k = i|k|$. For $E \leq E_{thr}$, there are no inelastic processes, and $|S_\ell| = 1$. k being purely imaginary, it follows from (2.4) that $|S_\ell^{(0)}| = 1$. Hence $|S_\ell^{(0)}| = 1$ for both real and imaginary k.

From the above it follows that $S_\ell^{(0)}$ can be written in the form $\exp\{2i\delta_\ell(k)\}$, with the phase $\delta_\ell(k)$ being real for both real and imaginary k, so that the expansion in powers of k contains only even powers of k.

Near the threshold, $kR \ll 1$, and we may neglect all powers of kR higher than the first, so that we have

$$S_0=e^{2i\delta_0}(1-kRa_0);\quad S_l=\exp(2i\delta_l)\ (l\neq 0), \tag{2.5}$$

where δ_ℓ denotes the phase of the ℓ-th partial wave at $E = E_\pi$ (i.e., k = 0). Hence near the threshold, all the S_ℓ $(\ell \neq 0)$ can be taken as energy independent, while S_0 is a linear function of k.

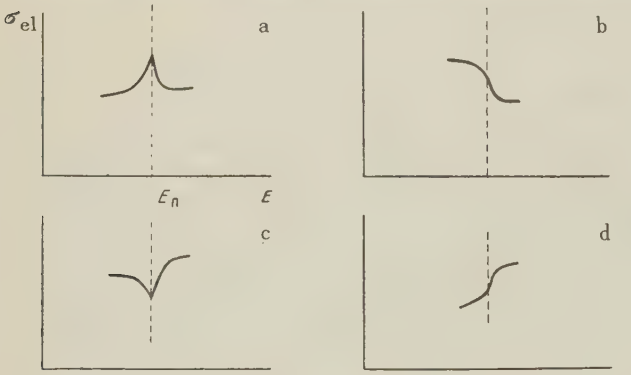

FIG. 1

Knowing the energy dependence of the S_ℓ, we can calculate the energy dependence of the elastic cross section near the threshold:

$$\sigma_{el}(\theta,E)=|f(\theta,E)|^2$$

$$\equiv\left|\frac{1}{2ik_1}\sum_l(2l+1)(S_l-1)P_l(\cos\theta)\right|^2=\sigma_{el}(\theta,E_{thr}) \tag{2.6}$$

$$-\frac{k_1}{2\pi}\sqrt{\sigma_{el}(\theta,E_{thr})\sigma_{inel}(|k|)}\begin{cases}\sin(2\delta_0-\alpha(\theta)) & \text{for } E>E_{thr}\\ \cos(2\delta_0-\alpha(\theta)) & \text{for } E<E_{thr}.\end{cases}$$

Here $\sigma_{el}(\theta, E_{thr})$ is the differential scattering cross section at $E = E_{thr}$, $\alpha(\theta)$ is the phase of the scattering amplitude at $E = E_{thr}$,

$$f(\theta,E_{thr})=e^{i\alpha(\theta)}\sqrt{\sigma_{el}(\theta,E_{thr})},$$

and $\sigma_{inel} = 2\pi a_0 |k| R/k_1^2$ is the total reaction cross section for $E > E_{thr}$.

Near the threshold, $\sigma_{inel}(|k|)$ varies with energy as $|k| \sim \sqrt{|E - E_{thr}|}$ (Ref. 2), while all the other quantities in (2.6) can be taken to be energy independent. Hence it follows from (2.6) that if $[2\delta_0 - \alpha(\theta)]$ is in the first quadrant, the elastic scattering cross section has a cusp at the threshold (Fig. 1a), if $[2\delta_0 - \alpha(\theta)]$ lies in the third quadrant, the peak turns into a valley (Fig. 1c), while if $[2\delta_0 - \alpha(\theta)]$ is in the second or fourth quadrants, the cross section exhibits a step (Figs. 1b, 1d).

As a function of $\sqrt{|E - E_{thr}|}$, σ_{el} varies linearly on both sides of the threshold. Knowing the slope before and after the threshold, and the cross section at threshold, we can use (2.6) to find the reaction cross section and the quantity $(2\delta_0 - \alpha)$. It is not difficult to see (cf. Sec. 3) that knowing $(2\delta_0 - \alpha)$ and the modulus of the scattering amplitude $f(\theta, E_{thr})$, we can easily obtain the full scattering amplitude and hence solve the phase analysis problem.

Formula (2.6) is applicable when $kR \ll 1$, which corresponds to an energy range of 20 – 50 kev around the threshold. This is also the width of the peak or step in the differential cross section. In deriving (2.6), it was assumed that there is no Coulomb interaction between b and Y. Hence all our results apply only to the case where one of the reaction products is a neutron.

In the above, the spins of all the particles were taken to be zero. However, the results can be generalized to the more interesting case of non-zero spins. This will be done in the next section.

3. THE CROSS SECTION FOR THE SCATTERING OF α PARTICLES FROM SPIN-ZERO NUCLEI NEAR THE THRESHOLD OF AN (α, n) REACTION

In this case, the spins of the initial particles a and X are zero, the spin of the neutron in ½, and the spin of the nucleus Y formed in the reaction is some half-odd-integer s.

Two situations are possible:

(a) $p(a)\,p(X)\,p(b)\,p(Y) = 1$ [$p(i)$ is the parity of particle i]. Near the threshold the particles b and Y have orbital angular momentum $\ell = 0$, i.e., a total momentum $I = s \pm 1/2$. Hence only that partial wave in the particles $(a + X)$ which has $\ell' = I$ (ℓ' even) can contribute to the reaction. Then the condition (2.2) will link M_0 with $S_{\ell'}$ rather than S_0, and (2.5) will be replaced by

$$S_{l'} = \exp(2i\delta_{l'})(1 - akR); \quad S_l = \exp(2i\delta_l) \quad (l \neq l'). \tag{3.1}$$

The scattering cross section will now be given by the formula

$$\sigma_{\text{el}}(\theta,\ E) = \sigma_{\text{el}}(\theta,\ E_{\text{thr}}) - \frac{k_1}{2\pi}\sqrt{\sigma_{\text{el}}(\theta,\ E_{\text{п}})}\,\sigma_{\text{inel}}(|k|)\,(2l'+1)\,P_{l'}(\cos\theta)\begin{cases}\sin(2\delta_{l'} - \alpha(\theta)) & E > E_{\text{thr}}\\ \cos(2\delta_{l'} - \alpha(\theta)) & E < E_{\text{thr}}\end{cases} \tag{3.2}$$

and we see that the anomaly in the cross section must disappear at angles such that $P_{\ell'}(\cos\theta) = 0$. From this one can find ℓ', and the spin s (up to an additive constant ± ½). We note that since ℓ' is even and $P_{2n}(\cos 90°) \neq 0$, the anomaly in the cross section does not vanish at $\theta = 90°$ (in the center of mass system).

(b) $p(a)\,p(X)\,p(b)\,p(Y) = -1$. The only difference between this case and the preceding one is that now ℓ' must be odd (by conservation of parity). The scattering cross section is given as before by (3.2), but since ℓ' is odd the anomaly in the cross section vanishes at $\theta = 90°$.

Hence from the energy dependence of the differential cross section for (α, α) scattering near the threshold of the (α, n) reaction one can find the following: (1) the cross section for the reaction (α, n), $\sigma_{\text{inel}}(|k|)$ (see Sec. 2), (2) the parity of the nucleus Y formed in the reaction, and (3) the spin of the nucleus Y (with an accuracy ± ½). In addition, one can find the quantity $(2\delta_{\ell'} - \alpha)$ by measuring the slope of the cross section before and after the threshold, and hence find all the elastic scattering phases δ_ℓ at threshold (the modulus $|f(\theta, E_{\text{thr}})|$ being known). In order to do this one must expand the experimentally known angular distribution in Legendre polynomials:

$$\exp\{i(\alpha - 2\delta_{l'})\}\,|f(\theta, E_{\text{thr}})| = \exp\{-2i\delta_{l'}\}\,f(\theta, E_{\text{thr}}) = \exp\{-2i\delta_{l'}\}\frac{1}{2ik_1}\sum_l (2l+1)(\exp\{2i\delta_l\} - 1)\,P_l(\cos\theta)$$

The coefficients in the expansion immediately determine the phases.

4. THE CROSS SECTION FOR THE SCATTERING OF α PARTICLES BY NUCLEI WITH SPIN ½ (OR PROTONS BY NUCLEI WITH SPIN ZERO) NEAR THE THRESHOLD OF THE (α, n) [OR (p, n)] REACTION

In this case, the spin of one of the initial particles (a) is ½, that of the other (X) is 0, the spin of particle b (neutron) is ½, while the spin of the nucleus Y formed in the reaction is some integer s. The non-zero spin of a introduces considerable complications. This is because in general the particles b + Y can have both spin $I = s + 1/2$ and $I = s - 1/2$. For a given ℓ in the incident wave, the particles a + X can have angular momentum $j = \ell \pm 1/2$, and for each momentum I of the particles b + Y, with arbitrary parity, there will always be a value of ℓ, with the appropriate parity, such that $\ell \pm 1/2 = I$. Hence near the threshold there will be two matrix elements depending linearly on k

$$S_{l'}^{s+1/2} = \exp\{2i\delta_{l'}^{s+1/2}\}(1 - a_1kR); \quad S_{l''}^{s-1/2} = \exp\{2i\delta_{l''}^{s-1/2}\}(1 - a_2kR); \quad S_l^j = \exp\{2i\delta_l^j\} \quad (j \neq s \pm 1/2), \tag{4.1}$$

If the intrinsic parity of the particles does not change during the reaction, then ℓ' and ℓ'' are even, while if the intrinsic parity does change, then ℓ' and ℓ'' are odd. Knowing the behavior of the S^j_ℓ, we can calculate the energy dependence of the scattering cross section X(aa)X and of the polarization $P(\theta, E)$ of the particle a:

$$\sigma_{el}(\theta, E) = |g(\theta, E)|^2 + |h(\theta, E)|^2; \qquad P(\theta, E) = 2\mathrm{Im}\,(h(\theta, E)\,g^*(\theta, E))/\sigma_{el}(\theta, E), \tag{4,2}$$

$$g(\theta, E) = \frac{1}{2ik_1}\sum_l [(l+1)(S_l^{l+1/2}-1) + l(S_l^{l-1/2}-1)]\,P_l; \quad h(\theta, E) = -\frac{1}{2ik_1}\sum_l [(S_l^{l+1/2} - S_l^{l-1/2})\,P_l^{(1)}. \tag{4.3}$$

If, for example, the intrinsic parity of the particles does not change during the reaction, and s is odd, then the following matrix elements depend linearly on k,

$$S_{l'}^{l'-1/2} = \exp\{2i\delta_{l'}^{l'-1/2}\}(1-a_1kR) \text{ and } S_{l''}^{l''+1/2} = \exp\{2i\delta_{l''}^{l''+1/2}\}(1-a_2kR); \quad l' = s+1; \quad l'' = s-1$$

and (4.3) becomes

$$\begin{aligned} g(\theta, E) &= g(\theta, E_{\text{thr}}) + \frac{ik}{2k_1}[a_1l'\exp\{2i\delta_{l'}^{l'-1/2}\}P_{l'} + a_2(l''+1)\exp\{2i\delta_{l''}^{l''+1/2}\}P_{l''}], \\ h(\theta_1 E) &= h(\theta, E_{\text{thr}}) - \frac{ik}{2k_1}\left[a_1\exp\{2i\delta_{l'}^{l'-1/2}\}P_{l'}^{(1)} + a_2\exp\{2i\delta_{l''}^{l''+1/2}\}P_{l''}^{(1}\right], \end{aligned} \tag{4.4}$$

where, as before, k_1 is the wave vector of the particle, and $P_\ell^{(1)}$ is an associated Legendre polynomial. Substituting (4.4) in (4.2), one easily obtains formulas for the cross section and polarization analogous to (3.2). As is to be expected, the scattering cross section and the polarization have singularities at the threshold. Near the threshold, $\sigma_{el}(\theta, E)$ and $P(\theta, E)$ are linear functions of k, so that experiments at a given scattering angle θ allow one to compute the following six quantities: $\sigma_{el}(\theta, E_{thr})$, $P(\theta, E_{thr})$ and the slopes of σ_{el} and P on both sides of the threshold. The analysis of the experimental data is not as simple as in the preceding case, but can still yield as before the spin and parity of the particle Y, the cross section $\sigma_{inel}^{s\pm 1/2}$ and also the phases δ^j_ℓ describing the elastic scattering.

That such an analysis can be carried out follows from the fact that at each angle the six quantities above depend only on the following six parameters: a_1, a_2, the phases and the moduli of the amplitudes $g(\theta, E_{thr})$ and $h(\theta, E_{thr})$. By their definitions, the quantities a_1 and a_2 do not depend on the scattering angle, so that the following method can be used to find the previously unknown orbital angular momenta ℓ' and ℓ''. Choosing values of ℓ' and ℓ'', we can solve, at each angle, the six equations given by the experiment for a_1, a_2 and the amplitudes g and h. If the values of a_1 and a_2 so obtained depend on the angle, new ℓ' and ℓ'' must be chosen and the process repeated until the true values of ℓ' and ℓ'' are found.

5. CONCLUDING REMARKS

With few changes, our method can be extended to cases where the scattered particles have higher spins. The resulting formulas are too complicated to write down in general, but all the qualitative results derive above still hold: the physical quantities characterizing the scattering (cross section, polarization, the correlation between polarizations) are singular at the threshold. Near the threshold, these quantities are linear functions of k. The energy dependence, near the threshold, of the scattering cross section and of the polarization give the spin and parity of the particle Y and also simplify the phase analysis considerably. To see how much the phase analysis is simplified, one need only note that experiments near the threshold give the phase shifts for the elastic scattering even when these are complex (i.e., there is absorption over and above the reaction whose threshold is considered); in general this is not possible to do without considering all the possible channels.

The reason that so much information can be gained from experiments near the threshold is that there one measures three quantities instead of only one (such as cross section, polarization, etc.). The three quantities are the absolute magnitude and its derivatives with respect to k on both sides of the threshold. This gives three times as many equations as usual for determining the unknowns.

In the preceding we have assumed that there is only one inelastic process X(ab) Y in addition to elasti scattering X(aa)X. Let us see what changes will be introduced if, for $E < E_{thr}$ there is a third reaction X(ac)Z in addition to elastic scattering X(aa)X. If the third reaction is described by matrix elements D_ℓ (for simplicity we take the spins to be zero), the conservation of particles for $E > E_{thr}$ can be written in the form

$$|S_l|^2+|D_l|^2+|M_l|^2=1 \tag{5.1}$$

near the threshold S_0 and D_0 will be linear functions of k:

$$S_0=S_0^{(0)}(1-a_1kR);\quad D_0=D_0^{(0)}(1-a_2kR). \tag{5.2}$$

It is easy to see that the reaction X(ac) Z decreases the size of the coefficient a_1 in the formula for S_0.

The singularity in the cross section which is associated with the threshold "distributes itself," in the case (5.1), over the scattering cross section for X(aa)X and the reaction cross section for X(ac) Z. Two conclusions follow: (1) near the threshold E_{thr} for the reaction X(ab)Y there are anomalies not only in the scattering cross section X(aa)X, but also in the cross sections for all the reactions X(ac) Z whose thresholds E_c lie below E_{thr}; (2) as the number of competing reactions X(ac) Z increases, the anomaly in the scattering cross section must decrease. How much it decreases depends on the details of the situation, but the decrease need not be small. However, the nature of the singularity (break in the cross section) does not change, and whether or not it can be discovered depends only on the precision of the experiment. This applies only when the threshold E_{thr} is sharp. If the state that Y is formed in is wide ($\Gamma > 50$ kev), then the anomaly in the cross section will be smeared out and it will not be possible to observe it.

How big should these effects be? From (2.6) it follows that the height of the peak (or step) in the scattering cross section is, in order of magnitude, equal to the total reaction cross section $\sigma_{inel}(k=1/R)$ at $k=1/R \approx 2\times 10^{12}$ cm, i.e., it is about 0.02 – 0.1 barns. Taking the differential cross section to be about 0.2 barns, we conclude that the height of the anomaly can be 10 – 50% of the cross section. This estimate agrees with the results quoted in Ref. 1 on the $Li^7(pp)Li^7$ scattering cross section.

All of the above results refer to the case that there is no Coulomb interaction between the particles b and Y. Only then is $M_0 \sim (kR)^{1/2}$. If b and Y repel each other electrostatically, then M_0 decreases exponentially as k goes to zero and no anomaly will be observed in the scattering cross section.

In conclusion, I wish to thank Ia. A. Sorodinsky and L. D. Puzikov for useful discussions.

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Translated by R. Krotkov
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THE INFLUENCE OF A He^3 IMPURITY ON THE VISCOSITY OF HELIUM II

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On the basis of the Landau theory of liquid helium, a determination is made of the coefficient of viscosity of a weak solution of He^3 in helium II as a function of temperature and concentration.

ACCORDING to the theory of Landau, a weak solution of He^3 in helium II can be regarded as a mixture of three gases of elementary excitations:[1] (1) a gas of rotons $\epsilon_r = \Delta_r + (p - p_0)^2/2m$, (2) a gas of phonons $\epsilon = sp$, and (3) a gas of impurity excitations $\epsilon_i = \Delta_i + p^2/2\mu$.

In order to solve the problem of the viscosity of the solution it is necessary to know how the elementary excitations interact with each other. The effective cross-sections for roton-roton, roton-phonon, and phonon-phonon scattering have been calculated by Landau and Khalatnikov[2] in their well known work on the theory of the viscosity of helium II. They have also discussed the question of the establishment of energy equilibrium in a phonon gas, which plays an important role in the study of kinetic effects in helium II. Effective cross-sections for the scattering of impurities by rotons and by each other, and for the scattering of phonons by impurities, have been calculated by Khalatnikov and the author[3] for an investigation of diffusion and of thermal conduction in solutions.

From the conservation laws it follows that, relative to the phonons, the rotons and the impurity excitations can be regarded as heavy particles. Moreover, the absolute values of the effective cross-sections for the scattering of the heavy particles by each other are much greater than the cross-sections for the scattering of phonons by these particles. These facts lead to the result that, for concentrations of He^3 atoms greater than $\sim 10^{-6}$ and for all temperatures, deviations of the roton and impurity distribution functions from their equilibrium values are determined only by processes which involve the scattering of the "heavy" excitations by each other. Furthermore, in solving the kinetic equations which determine the phonon distribution function, the rotons and impurities can be described by equilibrium Maxwellian distribution functions, since deviations of these functions from equilibrium are much smaller than deviations of the phonon distribution function from its equilibrium value. Both these circumstances make the calculations extremely simple and analogous to those carried out earlier in determining the thermal conductivity κ of a solution.[3]

The viscosity, as well as the thermal conductivity, of a solution is the sum of three parts: (1) the "roton viscosity" η_r, which depends on the transfer of momentum by rotons, (2) the "phonon viscosity" η_{ph}, which depends on the transfer of momentum by phonons, and (3) the "impurity viscosity" η_i, which depends on the transfer of momentum by the impurities. Analysis of the thermal conductivity of a solution of He^3 in helium II has shown that κ_r and κ_{ph}, the roton and phonon parts of the coefficient of thermal conductivity, differ from their corresponding values κ_{0r} and κ_{0ph} for pure helium only in that the scattering processes determining their effective times are modified by the presence of impurities. As we have shown,[3] κ_r is obtained from κ_{0r} by replacing the time t_{rr}^{-1} in the latter by the time $t_r^{-1} = t_{rr}^{-1} + t_{ri}^{-1}$, where t_{rr} and t_{ri} are quantities differing from the average times between roton-roton and roton-impurity collisions by temperature-independent factors of order unity. These times are[2,3]

$$t_{rr} = \hbar^4 / 4p_0 m \mid V_0 \mid^2 N_r = 0.7 \cdot 10^{-13} T^{-1/2} e^{\Delta_r/kT}, \tag{1}$$

$$t_{ri}^{-1} = \overline{v_i \sigma_{ir} N_i} = 3.3 \cdot 10^{12} c T^{1/2}. \tag{2}$$

Here the usual notation is used: Δ_r and p_0 are the zero-point energy and momentum of a roton, m the effective mass of a roton, N_r the number of rotons per unit volume, v_i the velocity of an impurity excitation, σ_{ir} the effective cross-section for the scattering of an impurity by a roton, N_i the number of impurity atoms per unit volume, $c = N_3 m_3/(N_3 m_3 + N_4 m_4)$ the concentration; M_3 and m_4, N_3 and N_4 the masses and numbers of atoms of He^3 and He^4 per unit volume. The representation of t_r^{-1} in the form of a sum is justified in the limiting cases $t_{rr} \ll t_{ri}$ and $t_{rr} \gg t_{ri}$, i.e., when rotons are scattered only by other rotons or only by impurities.

Because of the fact that the quantity t_{rr}^{-1}, which is proportional to the number of rotons, decreases exponentially with decreasing temperature, the transition from the case $t_{rr} \ll t_{ri}$ to the case $t_{rr} \gg t_{ri}$ (for a given concentration) will take place in a temperature interval of the order of ~ 0.1°K. Analogously, in the case of phonons, κ_{ph} will be obtained from κ_{0ph} by replacing θ_{phr}^{-1} by $\theta^{-1} = \theta_{phr}^{-1} + \theta_{phi}^{-1}$, where θ_{phr} and θ_{phi} are times characterizing the scattering of phonons by rotons and by impurities, respectively:[2,3]

$$1/\theta_{phr} = \frac{6!N_r}{4\pi s}\left[\frac{p_0 (kT/s)^2}{\hbar^2 \rho}\right]^2 A, \tag{3}$$

$$1/\theta_{phi} = \frac{6!N_i}{8\pi s}\left[\frac{(3\mu kT)^{1/2} (kT/s)^2}{\hbar^2 \rho}\right]^2 \delta. \tag{4}$$

Here s is the velocity of sound and ρ the density of helium. The quantities A and δ are temperature-independent constants of order unity.

The representation of θ^{-1} in the form of a sum $\theta_{phr}^{-1} + \theta_{phi}^{-1}$ is also justified only in the limiting cases $\theta_{phr} \ll \theta_{phi}$ and $\theta_{phr} \gg \theta_{phi}$, and the formula $\theta^{-1} = \theta_{phr}^{-1} + \theta_{phi}^{-1}$ represents an interpolation. Except for the replacement of t_{rr}^{-1} by t_r^{-1} and of θ_{phr}^{-1} by θ^{-1}, the form of the coefficients of thermal conductivity κ_{0r} and κ_{0ph} remains the same. It is, therefore, natural that the relation which exists between η_{0r} and η_{0ph}, on the one hand, and between κ_{0r} and κ_{0ph}, on the other, also remains the same for the corresponding coefficients for solutions of He^3 in helium II (this fact can be confirmed also by direct calculation). This relation was established by Khalatnikov[4] and has the form

$$\eta_{0r} = \varkappa_{0r} p_0^2 T / 5\Delta_r^2; \qquad \varkappa_{0r} = \Delta_r^2 t_{rr} N_r / 3mT, \tag{5}$$

$$\eta_{0ph} = \frac{T}{5s^2}\left(1 + \frac{\overline{\theta}_{phr}}{56\tau_{ph}}\right)^{-1}\left(1 - \frac{S_0 T}{\rho_{n0} s^2}\right)^{-1} \varkappa_{0ph}, \quad \varkappa_{0ph} = N_{ph} k \theta_{phr} s^2 \left(1 - \frac{S_0 T}{\rho_{n0} s^2}\right) \begin{cases} 19\,\dfrac{1 + 0.75\,\theta_{phr}/\theta_{ph}}{1 + 8\theta_{phr}/\theta_{ph}} & (\text{for } T > 0.9^\circ \text{K}) \\ 1.8 & (\text{for } T < 0.9^\circ \text{K}), \end{cases} \tag{6}$$

where τ_{ph} is a characteristic time for the phonon-phonon scattering process, which according to Landau and Khalatnikov,[2] is equal to

$$\tau_{ph}^{-1} = \frac{3 \cdot 13!\,(u + 2)^4}{5 \cdot 2^{19}\,(2\pi)^3\,\hbar^2 \rho^2 s}\left(\frac{kT}{s}\right)^9 \left(u = \frac{2\rho}{s}\frac{\partial s}{\partial \rho}\right). \tag{7}$$

Here $N_{ph} = 2.4 \times 4\pi\,(kT/2\pi\hbar s)^3$ is the number of phonons per unit volume, θ_{ph} is the characteristic time for the establishment of equilibrium with respect to the number of phonons in the phonon gas, and S_0 and ρ_{n0} are the entropy per unit volume and the normal density of pure helium. For the temperature region $T > 0.9$°K, $\theta_{phr}/56\,\tau_{ph} \ll 1$ and the second term in the parentheses in (6) can be dropped. The bar over the time θ_{phr} denotes that the time characterizing the scattering of phonons by rotons has been calculated with a different dependence of the deviations of the distribution function δn on the scattering angle than in the case of the thermal conductivity; namely, for calculations of the viscosity we have $\delta n \sim \cos\theta \sin\theta \cos\varphi$. In the case of a solution, τ_{ph}^{-1} can be set equal to zero for $c > 10^{-4}$. For concentrations $c < 10^{-4}$, $\theta/\theta_{ph} \ll 1$ for $T < 0.8$°K.

Taking all the above considerations into account and with the help of (5) and (6) we find the roton and phonon parts of the viscosity coefficient of a solution:

$$\eta_r = t_r p_0^2 N_r / 15m, \tag{8}$$

$$\eta_{ph} = \begin{cases} 3.8 \cdot N_{ph} k T \overline{\theta}\,\dfrac{1 + 0.75\overline{\theta}/\theta_{ph}}{1 + 8\,\theta/\theta_{ph}} \begin{cases} \text{for } T > 0.6^\circ \text{K};\ c > 10^{-4} \\ \text{for } T > 0.9^\circ \text{K};\ c < 10^{-4} \end{cases} \\ 0.35\,N_{ph} k T \overline{\theta}\left(1 + \dfrac{\overline{\theta}}{56\tau_{ph}}\right) \quad \text{for } T < 0.8^\circ \text{K};\ c < 10^{-4}. \end{cases} \tag{9}$$

Here account has been taken on the fact that to determine the viscosity coefficients from the thermal conductivity coefficients it is necessary to replace the unbarred effective times by those with a bar. Such a substitution has little effect on the effective times; thus, $\bar{\theta}_{phr}$ and $\bar{\theta}_{phi}$ differ from θ_{phr} and θ_{phi} only because of insignificant changes of the constants A in (3) and δ in (4), which remain of order of magnitude unity.

A calculation of the impurity part of the viscosity coefficient η_i can be carried out in the same way as in the kinetic theory of gases in the two limiting cases $t_{ir} \ll t_{ii}$ and $t_{ir} \gg t_{ii}$. Here t_{ir} is an effective time characterizing the scattering of impurities by rotons:[3]

$$t_{ir}^{-1} = \sigma_{ir} \sqrt{3kT/\mu}\, N_r = 1 \cdot 10^{13} T e^{-\Delta_r/kT}. \tag{10}$$

The effective time t_{ii} characterizing the scattering of impurities by each other is given by the equation

$$t_{ii}^{-1} = \overline{|v_{i1} - v_{i2}|\, \sigma_{ii} N} = \frac{a}{\pi} \frac{|V_{01}|^2 \mu^2}{\hbar^4} \sqrt{\frac{kT}{\pi\mu}}\, N = 2.9 \cdot 10^{11} a T^{1/2} c, \tag{11}$$

where μ is the effective mass of an impurity excitation, σ_{ii} is the effective cross-section for the scattering of impurities by impurities,[3] and V_{01} is a constant of the interaction between impurities. A constant a has been introduced in (11) since the magnitude of V_{01} is not known. A numerical value for (11) is obtained by taking $|V_{01}|^2 \sim 10^{-76}$ erg^2 cm^6. It can be assumed that the quantity a is on the order of unity. With the aid of the interpolation formula $t_i^{-1} = t_{ir}^{-1} + t_{ii}^{-1}$ we determine the effective time characterizing the scattering of impurities in the solution, and then after elementary calculations obtain

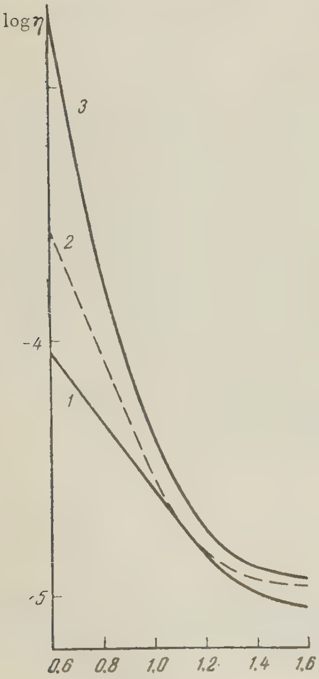
$$\eta = (15\pi/32)(kT/m_3)\, c\rho t_i. \tag{12}$$

After collecting the expressions obtained above, substituting numerical values for the parameters and making use of the final data given by Khalatnikov in his review article,[5] we obtain for the viscosity of the solution

$$\eta = \frac{1.15 \cdot 10^{-5}}{1 + 0.23 \cdot c e^{8.9/T}} + \frac{2.1 \cdot T^{1/2} \cdot 10^{-5}}{a + 2.9 \cdot 10 T^{1/2} c^{-1} e^{-8.9/T}} + \frac{3.75 \cdot 10^{-8} T^{-1/2} e^{8.9/T}}{1 + 3.5 \cdot 10^{-2} \delta T^{1/2} c e^{8.9/T}} \cdot \begin{cases} \dfrac{1 + 0.75\, \theta/\theta_{ph}}{1 + 8\, \theta/\theta_{ph}} & \left\{\begin{array}{l}\text{for } T \geqslant 0.6^\circ\text{K};\ c > 10^{-4} \\ \text{for } T > 0.9^\circ\text{K};\ c < 10^{-4}\end{array}\right. \\ 0.095\,(1 + \theta/56\tau_{ph})^{-1}, & \text{for } T < 0.8^\circ\text{K};\ c < 10^{-4}. \end{cases}$$

The temperature dependence of the viscosity for different concentrations is plotted in the figure. We see that above ~ 1.1°K the presence of impurities has slight effect in modifying the viscosity of pure helium II. Gradually, below 1.1°K, phonon-impurity scattering processes begin to play a predominant role, because of the abrupt decrease in the number of rotons, and lead to a sudden decrease in the phonon mean free path in the solution relative to its value in pure helium, and thus to a corresponding decrease in the viscosity of the solution relative to the viscosity of pure helium.

In conclusion we take advantage of the opportunity to express our gratitude to Professor I. M. Khalatnikov for discussions of this work.



Temperature dependence of the viscosity of a solution (in poise). Theoretical curves: (1) c = 10^{-2}; (2) c = 10^{-3}; (3) c = 0 (viscosity of pure helium II).

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 4 APRIL, 1958

THE USE OF LEONTOVICH'S BOUNDARY CONDITIONS IN THE THEORY OF CERENKOV RADIATION

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The field of a charge moving above a plane boundary is computed by using Leontovich's approximate boundary condition.

THE computation of the field in the case of the motion of charges or currents near the surface separating a medium from vacuum presents certain difficulties. It is therefore proposed in the present note to use in problems in which only the field in the vacuum is of importance the Leontovich approximate boundary condition[1]

$$\sqrt{\varepsilon'}E_t = \sqrt{\mu'}H_t, \tag{1}$$

where ϵ' and μ' are the dielectric constant and permeability of the isotropic medium,* while E_t and H_t are the tangential components of **E** and **H**.

We shall consider the case of greatest interest, when the transverse dimensions of the source are of the order of or greater than the dimensions of the source in the direction of motion. An evaluation of the corrections to (1) shows that the application of (1) to the plane boundary of separation is equivalent under these conditions to neglecting terms of the order of $1/\beta^2\mu'\epsilon'$, corresponding to velocities much higher than the velocity of light in the medium. This case is of particular interest in the generation of microwaves and for the stabilization of moving beams.

It should be noted that the condition (1) is satisfied with the accuracy indicated above far away from, as well as in the neighborhood of, the source. As an illustration we shall give the values of $\sqrt{\epsilon'}E_t/\sqrt{\mu'}H_t$ taken from two problems which have been solved exactly.

(a) In the case of uniform motion of a current filament along the separation boundary,[2]

$$\sqrt{\varepsilon'}E_t/\sqrt{\mu'}H_t = (1 - 1/\beta^2\varepsilon'\mu')^{-1/2} \approx 1 + 1/2\,\beta^2\varepsilon'\mu'.$$

(b) In the case of uniform motion of a point source normal to the surface of the dielectric[3]

$$\frac{\sqrt{\varepsilon'}E_t}{\sqrt{\mu'}H_t} = \left(1 + \frac{1-\beta^2}{\beta^2\varepsilon'\mu'}\right)^{-1/2} \approx 1 - \frac{1-\beta^2}{2\beta^2\varepsilon'\mu'}.$$

The use of condition (1) permits us to investigate problems in which only the field above the medium is of importance and the parameter $\beta^2\epsilon'\mu'$ is large. As an example we shall give the outline of the computation of the field of a point charge moving uniformly along the z axis at a distance $y = h$ from an arbitrary non-dispersive medium.

The boundary conditions $\Phi = 0$ and $\Psi = 0$ follow from (1) for the components of the potential A_z, A_y, φ where

$$\Phi = \sqrt{\mu'}A_y + \sqrt{\varepsilon'}\varphi, \quad \Psi = \sqrt{\mu'}\frac{\partial A_z}{\partial y} - \sqrt{\varepsilon'}\frac{1}{c}\frac{\partial A_z}{\partial t}. \tag{2}$$

According to Maliuzhinets[4] the functions Φ and Ψ can be found directly by the method of images and therefore determination of A_y, A_z, φ reduces to solving the system of equations (2) to which the Lorentz condition must also be added. Simple calculations yield

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*The case of anisotropic and gyrotropic media will be considered separately.

$$A_z = \frac{e\beta}{r} + e\beta\left[\frac{1-\lambda^2}{1+\lambda^2}\frac{1}{r^*} - 2\frac{\lambda}{\Lambda}W\right], \quad A_y = \frac{e\beta}{\theta}\left[\frac{2}{\varepsilon'-\mu'}\left(\mu'\frac{W_1}{\Lambda_1} - \varepsilon'\frac{W}{\Lambda}\right) + \frac{1}{r^*}\frac{2(\lambda_1+\lambda)}{\Lambda^2\Lambda_1^2}\right],$$

$$\varphi = e\left(\frac{1}{r} - \frac{1}{r^*}\right) - 2e\lambda_1\left[\frac{1}{r^*}\frac{\lambda_1+\lambda}{\Lambda^2\Lambda_1^2} + \frac{1}{\varepsilon'-\mu'}\left(\frac{\mu' W_1}{\Lambda_1} - \varepsilon'\frac{W}{\Lambda}\right)\right], \tag{3}$$

where

$$\xi = z - vt, \quad \theta^2 = 1 - \beta^2;$$
$$r^2 = x^2 + (y-h)^2 + (z-vt)^2; \ \lambda = (\beta/\theta)\sqrt{\varepsilon'/\mu'}, \ \Lambda^2 = 1 + \lambda^2;$$
$$r^{*2} = x^2 + (y+h)^2 + (z-vt)^2; \ \lambda_1 = (\beta/\theta)\sqrt{\mu'/\varepsilon'}, \ \Lambda_1^2 = 1 + \lambda_1^2;$$
$$W(\lambda) = \frac{\xi - \lambda(y+h)}{(\xi - \lambda(y+h))^2 + x^2\Lambda^2}\left(1 - \frac{(y+h)+\lambda\xi}{\Lambda r^*}\right); \quad W_1 \equiv W(\lambda_1).$$

The expression for the radiation losses obtained with the aid of (3) agrees for $\beta^2\mu'\epsilon' \to \infty$ with the exact expression.[5]

In the non-relativistic domain the solution of problems is further simplified by the possibility of expanding the field in powers of $\beta = v/c$.

$$\mathbf{E} = \mathbf{E}^{(0)} + \beta\mathbf{E}^{(1)} + \beta^2\mathbf{E}^{(2)} + \ldots, \quad \mathbf{H} = \mathbf{H}^{(0)} + \beta\mathbf{H}^{(1)} + \beta^2\mathbf{H}^{(2)} + \ldots$$

For the sake of definiteness we shall consider the motion of a charge near a certain boundary over which the condition (1) is satisfied. In this case it may be shown[6] that

$$E_t^{(0)} / E_n^{(0)} \sim \beta\sqrt{\mu'/\varepsilon'} = \lambda_1. \tag{4}$$

From (4) it follows that for $\lambda_1 \gg 1$ the electric field is "expelled" from the medium, while for $\lambda_1 \ll 1$ it enters the medium essentially at right angles. This permits one to take as the basis the solution of Laplace's equation with the boundary condition $E_n^{(0)} = 0$ or $E_t^{(0)} = 0$ and then by using (1) to find $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$, and thus to compute the intensity of radiation. The approximation obtained for $\lambda_1 \gg 1$ will correspond to expanding the intensity of radiation in powers of $1/\lambda_1$, while for $\lambda_1 \ll 1$ the expansion is carried out with respect to $1/\lambda$. With this accuracy one can easily find, for instance, the field of a bunch of charges moving at an arbitrary angle to the surface of a dielectric ($\epsilon' \gg \mu'$). This field is of the form

$$\varphi = \varphi_0 - \varphi_s - 2\frac{v_y}{c}\sqrt{\mu'/\varepsilon'}\varphi_s, \quad A_x = 0, \quad A_y = -2\sqrt{\mu'/\varepsilon'}\varphi_s + \frac{v_y}{c}(\varphi_0 + \varphi_s) + \sqrt{\frac{\mu'}{\varepsilon'}}v_y\int_{-\infty}^{t}\frac{\partial\varphi_s}{\partial y}dt,$$

$$A_z = \frac{v_z}{c}(\varphi_0 - \varphi_s) - 2v_z\sqrt{\frac{\mu'}{\varepsilon'}}\int_{-\infty}^{t}\frac{\partial\varphi_s}{\partial y}dt,$$

where φ_0 is the potential of the bunch in homogeneous space, while φ_s is the potential of its image taken with opposite sign.

In the case when the motion takes place near a curved surface the error due to the use of (1) will be of the order of

$$h(1/R_1 - 1/R_2)/\beta\sqrt{\mu'\varepsilon'},$$

where R_1 and R_2 are the principal radii of curvature, while h is the distance of the moving source from the surface. In the non-relativistic case one may also use simpler boundary conditions as was done above in the case of the plane boundary. We have calculated by this method the instantaneous value of the flux of the Poynting vector into a dielectric cylinder ($\lambda_1 \ll 1$, $\lambda \gg 1$) of radius a when a charged filament parallel to its axis passes over it. This flux turned out to be equal to

$$P = 2c\sqrt{\frac{\mu'}{\varepsilon'}}\rho^2\frac{\beta^2}{a}\frac{2r^2 + (r^2-1)p^2}{r^2(r^2-1)}.$$

where pa is the impact distance, $ar = \sqrt{a^2p^2 + (vt)^2}$ is the distance from the filament to the axis of the cylinder, and ρ is the charge density of the filament.

I am grateful to Prof. A. A. Sokolov and to Prof. Kh. Ia. Khristov for their interest in this work.

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THERMODYNAMICS OF THE He I — He II PHASE TRANSITION IN HELIUM ISOTOPE MIXTURES

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A thermodynamic investigation of the phenomena associated with a phase transition of the second order in liquid mixtures is carried out. The results obtained are compared with the experimental data on liquid-vapor equilibrium in the $He^3 - He^4$ system.

In pure He^4, as is well-known, the He I — He II transition is a phase transition of the second order. The question of the nature of this transition in mixtures of the helium isotopes has been repeatedly submitted to theoretical investigation, as a result of which, in particular, it has been predicted[1] that the λ-transition in such mixtures will be a transition of the first order for sufficiently small concentrations and high temperatures.

Inasmuch as this problem is of fundamental significance, it appears desirable to carry out a detailed thermodynamic investigation of the phenomena associated with the He I — He II transition in helium isotope mixtures, and to compare the results thus obtained with the experimental data (cf. also references 2, 3).

It is necessary first of all to note that a conclusion regarding the nature of the He I — He II phase transition in helium isotope mixtures can be drawn from a study of the dependence of the vapor pressures of these mixtures upon their He^3 content. In the case of a first order phase transition, in fact, there should be observed a separation of the liquid phase into two mixtures of differing He^3 content, which would yield vapor pressures independent of the concentration of the light isotope in the region of separation.

Carefully-conducted experiments[4-6], however, have shown that in the interval 1.35 — 3.0°K there is a marked dependence of the vapor pressure upon the He^3 content of the liquid phase, indicating that the hypothesis concerning the occurrence of a first order phase transition in this temperature interval is not borne out.

Thus, as in the case of pure He^4, the He I — He II phase transition in helium isotope mixtures within the temperature range investigated is a phase transition of the second order.*

Inasmuch as this is the only instance of the occurrence of a second order phase transition in liquid mixtures, the opportunity provided for investigating the special features arising therefrom is extremely interesting. In particular, it is possible on purely thermodynamic grounds to draw conclusions concerning the special features appearing on the $P-T$ curves at the transition through the λ-point, and concerning the behavior of the heats of mixing and vaporization at the transition through the same temperature.

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*It must be noted that Walters and Fairbank[7] have recently observed a separation of helium isotope mixtures into two phases below 0.8°K.

Use of the experimental data which have been obtained[4-6] relative to liquid-vapor equilibrium in the He^3-He^4 system makes possible a direct comparison of the theoretical results with experiment.

1. FEATURES OF THE LIQUID-VAPOR DIAGRAM IN THE VICINITY OF THE λ-POINT

Let us represent the thermodynamic potential for a mixture in the form $\Phi = N\varphi$, where φ is the thermodynamic potential per particle and $N = N_3 + N_4$ is the total number of particles. Representing the concentration of the mixture by $x_L = N_3/N$, we find the chemical potentials μ_3^m and μ_4^m of the components in the mixture:

$$\mu_3^m = \varphi + (1 - x_L)\frac{\partial\varphi}{\partial x_L}, \quad \mu_4^m = \varphi - x_L\frac{\partial\varphi}{\partial x_L}. \tag{1}$$

Using the condition of equilibrium between the phases and the expressions for the chemical potentials of the constituents in the gaseous phase, we obtain, assuming ideality for the vapor:

$$kT\ln P_3 + \chi_3(T) = \varphi + (1 - x_L)\,\partial\varphi/\partial x_L, \quad kT\ln P_4 + \chi_4(T) = \varphi - x_L\,\partial\varphi/\partial x_L, \tag{2}$$

where P_3 and P_4 are the partial pressures.

Taking the total derivative with respect to temperature on both sides of Eqs. (2), we arrive at the expressions:

$$-S_3 + kT\frac{d}{dt}\ln P_3 = \frac{\partial\varphi}{\partial T} + (1 - x_L)\frac{\partial^2\varphi}{\partial T\partial x_L}, \quad -S_4 + kT\frac{d}{dT}\ln P_4 = \frac{\partial\varphi}{\partial T} - x_L\frac{\partial^2\varphi}{\partial T\partial x_L}, \tag{3}$$

where S_3 and S_4 are the partial entropies per particle. At the temperature T_λ, inasmuch as at a phase transition of the second order the first derivative of the thermodynamic potential suffers a break in slope, and the second derivative a discontinuity, we obtain:

$$kT_\lambda\Delta\frac{d\ln P_3}{dT} = (1 - x_L)\,\Delta\frac{\partial^2\varphi}{\partial T\,\partial x_L}, \quad kT_\lambda\Delta\frac{d\ln P_4}{dT} = -x_L\Delta\frac{\partial^2\varphi}{\partial T\partial x_L}, \tag{4}$$

where

$$\Delta f = f(T_\lambda - 0) - f(T_\lambda + 0).$$

From (2) and (4) it follows immediately that in this case at T_λ there should be observed breaks in the P_3-T and P_4-T curves, and discontinuities in the $dP_3/dT-T$ and $dP_4/dT-T$ curves.

Taking into account the fact that near the transition point[8]

$$\varphi_{II} = \varphi_I - (\Delta C_p/2T_\lambda)(T - T_\lambda)^2, \tag{5}$$

where φ_{II} and φ_I are the values of the thermodynamic potential below and above the λ-point, respectively, and ΔC_p is the discontinuity in the specific heat at T_λ, we obtain

$$\Delta(\partial^2\varphi/\partial T\partial x_L) = (\Delta C_p/T_\lambda)\,\partial T_\lambda/\partial x_L, \tag{6}$$

and, in place of (4)

$$kT_\lambda\Delta\frac{d\ln P_3}{dT} = (1 - x_L)\frac{\Delta C_p}{T_\lambda}\frac{\partial T_\lambda}{\partial x_L}, \quad kT_\lambda\Delta\frac{d\ln P_4}{dT} = -x_L\frac{\Delta C_p}{T_\lambda}\frac{\partial T_\lambda}{\partial x_L}. \tag{7}$$

Using (7), it is possible to find a relation between the discontinuities in the derivatives of the partial pressures

$$x_L\Delta(d\ln P_3/dT) + (1 - x_L)\,\Delta(d\ln P_4/dT) = 0. \tag{8}$$

Setting

$$P_3 = x_V P; \quad P_4 = (1 - x_V) P, \tag{9}$$

where x_V is the concentration of He^3 in the vapor, we obtain from (7) an expression for the discontinuity in the derivative of the total pressure with respect to temperature

$$kT_\lambda \Delta \frac{d \ln P}{dT} = (x_V - x_L) \frac{\Delta C_p}{T_\lambda} \frac{\partial T_\lambda}{\partial x_L}. \tag{10}$$

Since $\partial T_\lambda / \partial x_L < 0$,[3] $\Delta C_p > 0$,[8] and $x_V > x_L$, $\Delta(d \ln P/dT) < 0$, which is found to be in agreement with experiment.[3]

Thus at a second order phase transition in mixtures the derivatives with respect to temperature of the partial and total pressures suffer discontinuities. It should be remarked, however, that magnitude of the discontinuity in the derivative of the total pressure is small, as is evident from (10); this is a consequence of the fact that the discontinuities in the derivatives of the partial pressures have opposite signs and the same order of magnitude [cf. Eqs. (4) and (8)]. As a result of this and also of the large value of dP/dT, the comparatively small discontinuity in this quantity at the λ-point will be scarcely discernible.

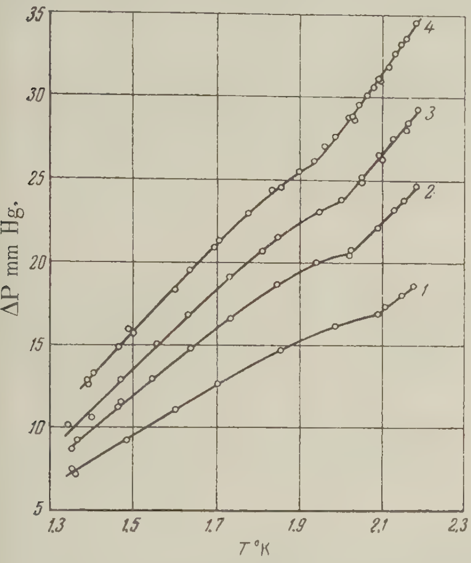

FIG. 1. Dependence of the difference between the mixture and He^4 vapor pressures (ΔP) upon temperature: (1) 8.3% He^3; (2) 11.1% He^3; (3) 13.4% He^3; (4) 16.7% He^3.

The dependence of ΔP upon T, where ΔP is the difference between the vapor pressures of the mixture and of pure He^4, has been investigated experimentally, and in this case breaks at T_λ are clearly visible. This is explained by the fact that the magnitude of $d(\Delta P)/dT$ has now been considerably reduced, while the magnitude of the discontinuity has remained the same. The dependence of ΔP upon T[3,6] is represented in Fig. 1 for mixtures of varying He^3 content. On each curve a break is visible at T_λ. This confirms the conclusions we have drawn regarding the characteristics of the $P - T$ curves, and provides a new possibility for determination of the displacement of T_λ with increasing He^3 concentration in the mixture.[3]

It is essential to note that at the λ-point a discontinuity should be observed in the first derivative with respect to temperature of the He^3 concentration in the gaseous phase (x_V). Specifically, substituting the values for P_3 and P_4 from (9) into (3), we readily find

$$\Delta \frac{\partial x_V}{\partial T} = x_V(1 - x_V) \frac{\Delta C_p}{kT_\lambda^2} \frac{\partial T_\lambda}{\partial x_L}; \tag{11}$$

$$\Delta \frac{\partial}{\partial T}\left(\ln \frac{c_V}{c_L}\right) = \frac{\Delta C_p}{kT_\lambda^2} \frac{\partial T_\lambda}{\partial x_L}, \tag{12}$$

where

$$c_V = (N_3/N_4)_V \quad c_L = (N_3/N_4)_L$$

Thus there follows from Eq. (11) the presence of a discontinuity in the derivative of x_V with respect to temperature, and from Eq. (12), the existence of a discontinuity in the derivative of the distribution coefficient with respect to temperature.*

The occurrence of breaks in the $P - T$ and $x_V - T$ curves at T_λ should lead to breaks in the two curves of the liquid-vapor equilibrium diagram. It should, however, be noted that these breaks do not appear in the published diagram,[6] as a consequence of the fact that the variation in these quantities at the λ-point is small, and also because the function $x_V(T)$ was determined, not directly, but from the diagram.

The magnitude of the discontinuity in the derivative of the total pressure with respect to x_L at the transition temperature is

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*This expression was obtained by us in 1950.[2] More recently it has been obtained by de Boer and Gorter.[9]

$$\Delta(\partial \ln P / \partial x_L) = (x_L - x_V)(\Delta C_P / kT_\lambda^2)(\partial T_\lambda / \partial x_L)^2. \tag{13}$$

It is evident from (13) that the discontinuity in the derivative of the pressure with respect to the concentration of the liquid phase is negative.

2. THE HEAT OF MIXING AND ITS DEPENDENCE UPON TEMPERATURE AND CONCENTRATION

At the temperature of the He I — He II phase transition there should also be observed a discontinuity in the heat of mixing. The magnitude of this discontinuity can be computed using an expression[8] for the quantity of heat (q_3) absorbed during the transfer of a single particle out of the pure liquid substance (He^3) into the mixture:

$$q_3 = \frac{\partial W^m}{\partial N_3^m} - W_3^L = -kT^2 \frac{\partial}{\partial T} \ln \frac{P_3}{P_3^0}, \tag{14}$$

where $\partial W^m / \partial N_3^m$ is the change in the heat function of the mixture as one atom of He^3 is added to it, while W_3^L is the molecular heat function for pure liquid He^3.

At T_λ

$$\Delta q_3 = \Delta(\partial W^m / \partial N_3^m),$$

since

$$\partial W^m / \partial N_3^m = \partial \Phi / \partial N_3^m - T \partial^2 \Phi / \partial T \partial N_3^m.$$

Therefore

$$\Delta q_3 = -T_\lambda \Delta(\partial^2 \Phi / \partial T \partial N_3^m).$$

Taking the fact that $\Phi = N\varphi$ and Eq. (6) into account, we obtain

$$\Delta q_3 = -(1 - x_L)\Delta C_P \partial T_\lambda / \partial x_L. \tag{15}$$

In the case of small concentrations, using the value $\Delta C_P = 7$ cal/mole degree for He^4, from Ref. 10, and the value of $\partial T / \partial x_L = -1.5°$, from Ref. 3, we obtain for the discontinuity in the heat of mixing at the He II — He I transition $\Delta q_3 = 10.5$ cal/mole.

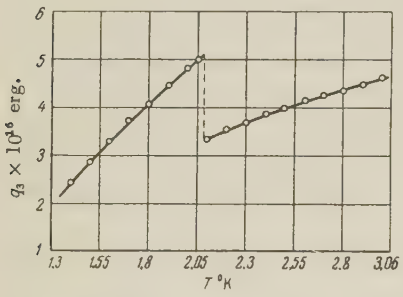

FIG. 2. Dependence of the heat of mixing (q_3) upon temperature. Calculated data for a mixture with 8.3% He^3.

It should be noted that the indicated changes in the distribution coefficient and the heat of mixing at the λ-point can be related to the discontinuities in the derivatives of the partial and total pressures:

$$\Delta \frac{\partial}{\partial T}\left(\ln \frac{c_V}{c_L}\right) = \frac{1}{x_V - x_L} \Delta \frac{d \ln P}{dT}, \qquad \Delta q_3 = -kT_\lambda^2 \frac{1 - x_L}{x_V - x_L} \Delta \frac{d \ln P}{dt} \tag{16}$$

Using the experimental data for the dependence of the vapor pressure upon temperature for mixtures of varying He^3 content, it would be possible to compute both the one quantity and the other from (16). In view, however, of the great difficulty in determining sufficiently precisely the indicated discontinuities in the derivatives of the partial and total pressures with respect to temperature, it does not appear possible as yet to use this method to compute the quantities in (16).

The temperature dependence of q_3 to either side of T_λ can be computed using Eq. (14). The results of such a calculation are presented in Fig. 2 for a mixture of 8.3% He^3 content. The error in the magnitude of the discontinuity Δq_3 at the λ-point is naturally determined here by the error in determining the magnitude of the discontinuity in the derivative of P_3 with respect to temperature.

3. THE HEAT OF TRANSITION AND ITS DEPENDENCE UPON He^3 CONTENT AND TEMPERATURE

Using the liquid-vapor diagram of state which has already been obtained,[6] it is possible to determine the temperature and concentration dependences of the heat of the transition from mixture to vapor. For the case of mixtures, however, the determination of the heat of transition itself, as is well known, requires some elucidation.

The fact is that the relative quantities of liquid and vapor are in this case related by the lever rule

$$N_V / N_L = (x_L - c) / (c - x_V), \tag{17}$$

where

$$c = (N_V^{(3)} + N_L^{(3)}) / (N_V + N_L)$$

is the mean concentration of the He^3 isotope in the system. Therefore the transfer of a molecule from the liquid to the vapor (or vice versa) by an equilibrium process (in accordance with the state diagram) cannot take place without variation of the pressure or the temperature. Such a situation does not arise in a single-component system, in which the transfer takes place at a point (for constant P and T).

Thus in determining the heat of transition it is necessary to specify the process by which the transition takes place. It is natural to investigate two processes: a) the transition takes place at constant temperature (T = const.), b) the transition takes place at constant pressure (P = const.).

Inasmuch as the heat of transition from gas to liquid for a single particle is

$$\lambda = T \delta S / \delta N_V, \tag{18}$$

it is necessary, in order to determine this, to compute the change in entropy δS accompanying the transfer of δN_V particles from the gas to the liquid. In doing this it is necessary to remember that the variation of all the parameters of the system proceeds in accordance with the diagram of state.

If it is assumed that T = const. [case (a)], then, taking into account the constancy of the number of particles of each sort in the system, and the fact that P and x_V are in this case functions of x_L, we readily obtain

$$\delta S / \delta N_V = S_V - S_L + \frac{x_L - x_V}{1 + \frac{x_L - c}{c - x_V} \frac{\partial x_V}{\partial x_L}} \left\{ \frac{\partial S_L}{\partial P} \frac{\partial P}{\partial x_L} + \frac{\partial S_L}{\partial x_L} + \frac{x_L - c}{c - x_V} \left(\frac{\partial S_V}{\partial P} \frac{\partial P}{\partial x_L} + \frac{\partial S_V}{\partial x_V} \frac{\partial x_V}{\partial x_L} \right) \right\}. \tag{19}$$

Here S_V and S_L are the entropies per particle of the gas and the liquid. In deriving Eq. (19) the lever rule (17) was employed.

If $x_L = c$ (Fig. 3; a large quantity of liquid and virtually no gas in the system), then

$$\frac{\delta S}{\delta N_V} = (S_V - S_L) + (x_L - x_V) \left(\frac{\partial S_L}{\partial P} \frac{\partial P}{\partial x_L} + \frac{\partial S_L}{\partial x_L} \right). \tag{20}$$

If $x_V = c$ (Fig. 4; a large quantity of gas and virtually no liquid in the system), then

$$\frac{\delta S}{\delta N_V} = (S_V - S_L) + (x_L - x_V) \left(\frac{\partial S_V}{\partial P} \frac{\partial P}{\partial x_V} + \frac{\partial S_V}{\partial x_V} \right). \tag{21}$$

Making use of the fact that

$$S_V = x_V S_V^{(3)} + (1 - x_V) S_V^{(4)}, \quad S_L = x_L S_L^{(3)} + (1 - x_L) S_L^{(4)}, \tag{22}$$

as well as of the equality of the chemical potentials of each component in the gas and the liquid, we find from (20), (21), and (18), assuming the gas to be ideal: for $x_L = c$:

$$\lambda_{T, x_L} = kT^2 \left(\frac{\partial \ln P}{\partial T} \right)_{x_L} + (x_L - x_V) T \frac{\partial S_L}{\partial P} \left(\frac{\partial P}{\partial x_L} \right)_T; \tag{23}$$

for $x_V = c$:

$$\lambda_{T, x_V} = kT^2 \left(\frac{\partial \ln P}{\partial T} \right)_{x_V} + (x_L - x_V) T \frac{\partial S_V}{\partial P} \left(\frac{\partial P}{\partial x_V} \right)_T. \tag{24}$$

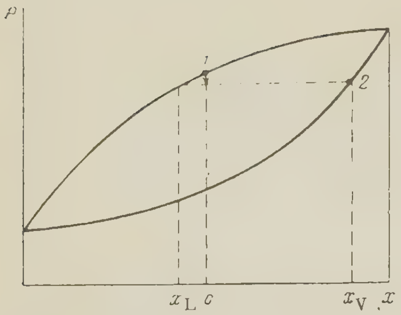

FIG. 3. For calculation of the heat of transition in the case T = const. and c = x_L: (1) state of the system before the transition; (2) concentration in the gaseous phase formed as a result of the transition; the arrow indicates the motion of the system due to the transition.

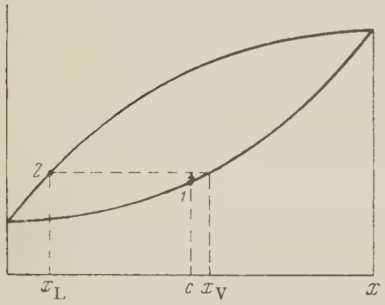

FIG. 4. For calculation of the heat of transition in the case T = const. and c = x_V: (1) state of the system before the transition; (2) concentration in the liquid phase formed as a result of the transition; the arrow indicates the motion of the system due to the transition.

If we take into account the fact that the entropy of the liquid is practically independent of the pressure, and if in calculating the second term in (24) we assume that the gas is ideal, we obtain:

$$\lambda_{T,x_L} = kT^2 (\partial \ln P/\partial T)_{x_L}, \tag{25}$$

$$\lambda_{T,x_V} = kT^2 \left(\frac{\partial \ln P}{\partial T}\right)_{x_V} + kT(x_V - x_L)\left(\frac{\partial \ln P}{\partial x_V}\right)_T. \tag{26}$$

In calculating the second term in Eq. (26) it is convenient to make use of the expression[8]

$$(\partial \ln P/\partial x_V)_T = (x_V - x_L)/x_V(1 - x_V). \tag{27}$$

Then

$$\lambda_{T,x_V} = kT^2 (\partial \ln P/\partial T)_{x_V} + kT(x_V - x_L)^2/x_V(1 - x_V). \tag{28}$$

In precisely the same way it is possible to calculate the corresponding heats for the case P = const.:

$$\lambda_{P,x_L} = kT^2 (\partial \ln P/\partial T)_{x_L} + (x_L - x_V)\, C_P^L (\partial T/\partial x_L)_P; \tag{29}$$

$$\lambda_{P,x_V} = kT^2 (\partial \ln P/\partial T)_{x_V} + (x_L - x_V)\, C_P^V (\partial T/\partial x_V)_P, \tag{30}$$

where C_P^L and C_P^V are the specific heats per particle of the liquid and the vapor.

Analysis of the expressions thus obtained shows that for each of the cases investigated [Eqs. (25), (28) — (30)] the heat of transition is composed of two terms,* the first representing the heat entering directly into the transfer of the particle from one phase into the other, and the second representing the heat associated with the change in entropy of the system due to this transfer. We note that in a real experiment (under the corresponding equilibrium conditions) the heats measured will be just those calculated here. The possibility of separating each of the heats into two components, each having a definite physical significance, arises from the fact that the first component may be obtained directly, from the quantity of heat absorbed during the transfer from the solution into the vapor of a single particle of the first constituent (λ_3), and from the corresponding quantity for the second constituent (λ_4). In actuality, when one mole of the mixture passes from the liquid, of concentration x_L, into the gaseous phase, its concentration will be x_V; in consequence $x_V\lambda_3 + (1 - x_V)\lambda_4$ calories (as calculated for a single particle) will be absorbed in the process. In accordance with reference 8, however, this quantity will be equal to the first term

$$x_V\lambda_3 + (1 - x_V)\lambda_4 = kT^2 (\partial \ln P/\partial T)\, x_L. \tag{31}$$

The heat of transition for the case of a process taking place at T = const. can be calculated for the cases $x_L = c$ and $x_V = c$ from (25) and (28), using the data obtained from investigation of liquid-vapor equilibrium in the $He^3 - He^4$ system. The results of these calculations are presented in Fig. 5 ($x_L = c$), which shows the dependence of λ_{T,x_L} upon the concentration of the phase being formed (x_V). It should be mentioned that in calculating λ_{T,x_L} the non-ideality of the vapor was taken into account by determining the molar volume using the second virial coefficient.[11] The values of λ for pure He^4 and He^3 were taken from Refs. 12 and 13. The solid curves in Fig. 5 refer to various temperatures. The dashed lines

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*The absence of a second term from Eq. (25) is of course due to the fact that we have ignored the variation of the liquid phase entropy with pressure.

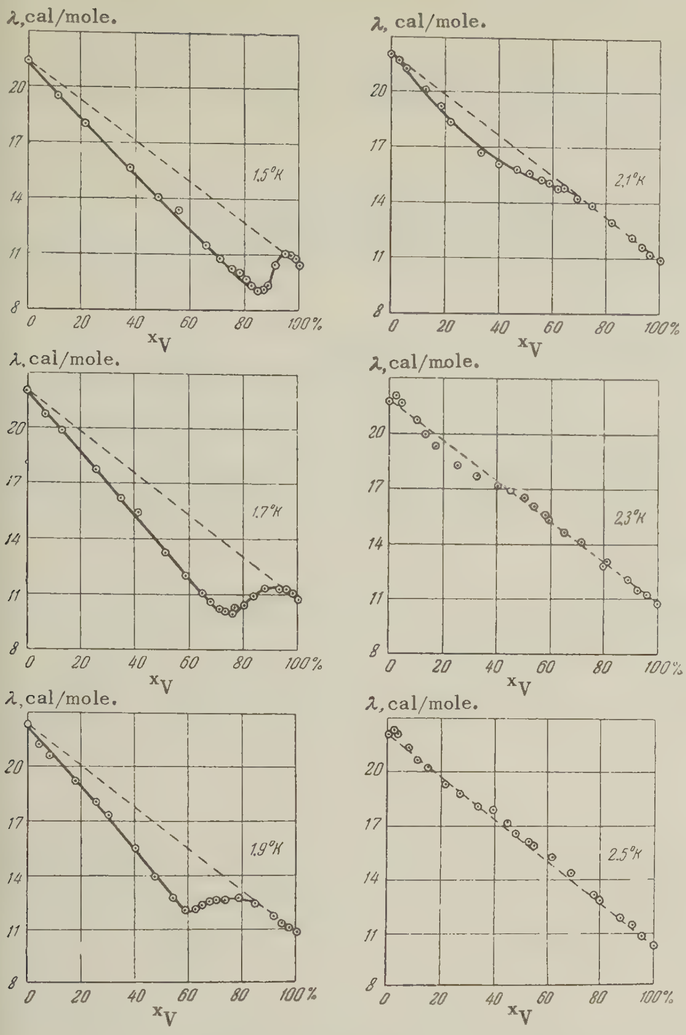

FIG. 5. Dependence of the heat of transition λ_{T,x_L} upon the concentration of the phase being formed (x_V).

show the heat of transition which would obtain in the additive case.

From these diagrams it follows that the heat of transition as calculated by the method indicated passes through a minimum, the position of which moves in the direction of smaller x_V as the temperature is raised. In the region of high x_V the points lie on the dashed line corresponding to the additive case, while the concentration region over which this is true increases as the temperature rises. Above 2.2°K λ_{T,x_L} has a practically linear dependence upon x_V.

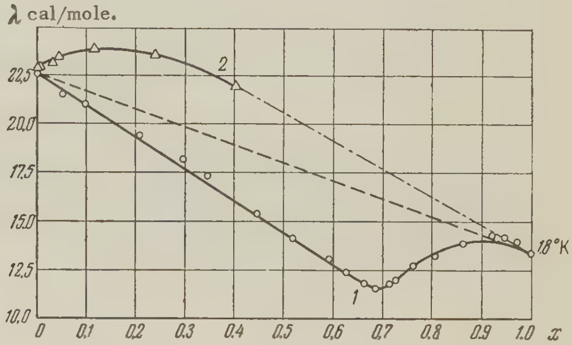

FIG. 6. Dependence of the heats of transition λ_{T,x_L} (curve 1) and λ_{T,x_V} (curve 2) upon the concentration of the phase being formed.

Fig. 6 shows for a temperature of 1.8°K the dependence of the heats of transition λ_{T,x_L} (curve 1) and λ_{T,x_V} (curve 2) upon the concentration of the phase being formed (x_V in the first case and x_L in the second), as calculated from Eqs. (25) and (28).

It is clearly evident that these heats differ fundamentally from one another, both in magnitude in the nature of their dependence upon the concentration.

It should be noted that at the transition through the λ-point the heat of transition suffers a discontinuity, as is evident from Eqs. (25), (28), (29), and (30). Due to the small magnitude of the change in dP/dT, such a discontinuity is not clearly displayed in the calculated heats. It can be supposed, however, that the considerable change in the heat of transition near the λ-point which is evident in Fig. 5 is connected with this circumstance.

CONCLUSIONS

1. A thermodynamic study of the phenomena associated with the He I — He II transition in mixtures of the helium isotopes has been carried out. It is shown that in the interval 1.35 — 3.0°K this transition is a phase transition of the second order.

2. It is shown that at the temperature of a second order phase transition in helium isotope mixtures there should be observed discontinuities in the derivatives of the partial and total pressures with respect to temperature. This effect is confirmed experimentally.

3. It is established that at T_λ there should be observed a discontinuity in the derivative of the distribution coefficient with respect to temperature.

4. It is shown that at T_λ there should be observed a discontinuity in the heat of mixing and a discontinuity in the heat of vaporization. A numerical value for the magnitude of the discontinuity in the heat of mixing has been obtained for weak mixtures. For one of these mixtures the temperature dependence of the heat of mixing has been calculated. The dependence of the heat of transition upon the concentration of the phase being formed has been calculated for various temperatures.

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 4 APRIL, 1958

ON THE EQUATIONS OF THE OSCILLATIONS OF THE MOLECULAR GENERATOR

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By use of a density matrix in the proper approximation, equations are derived that describe both stationary and nonstationary operating conditions of the molecular generator.

THE theoretical problems connected with the molecular generator have been discussed by several authors.[1-3] In these articles, however, only stationary or nearly stationary operating conditions are described. It is of interest to derive equations of the molecular generator that are valid for arbitrary operating conditions.

Upon replacing the resonant cavity by an equivalent circuit it is not difficult to derive the following equation, which describes the oscillations in the circuit:

$$\ddot{E} + 4\pi\ddot{P} + (\omega_0/Q)(\dot{E} + 4\pi\dot{P}) + \omega_0^2 E = 0. \tag{1}$$

Here E is the electric field intensity, P is the polarization of the medium in the resonant cavity, Q is the figure of merit of the circuit, and ω_0 is the natural frequency of the circuit. Equation (1) is still insufficient to describe the operation of the molecular oscillator. It remains to find equations that connect E with P.

In Refs. 1 to 3 and others it is assumed that E depends on the time harmonically (with frequency ω), and in this case the relation between E and P is given in a certain approximation. However, because the relation between E and P is nonlinear (saturation effect), summation over all frequencies ω does not lead to a general relation between E and P. The goal of the present work is to find such a relation.

Molecules in the presence of an alternating electric field E(t), like any other quantum-mechanical system, are described by a density matrix $\hat{\rho}$. The latter in our case obeys the equation

$$i\hbar\partial\hat{\rho}/\partial t = H\hat{\rho} - \hat{\rho}H, \tag{2}$$

where

$$H = H_0 - \hat{\mu}E, \tag{3}$$

H_0 is the Hamiltonian of the free (without field) molecule, and $\hat{\mu}$ is the operator for the dipole moment of the molecule.

The polarization, i.e., the average dipole moment of the gas per unit volume, is given by

$$P = \operatorname{Sp}(\hat{\rho}\hat{\mu}). \tag{4}$$

Here $\hat{\rho}$ is referred to unit volume. At a given instant of time t the contribution to the polarization P (as well as to $\hat{\rho}$) comes from those molecules which entered the resonator at a time $t_0 = t - \theta$; the largest contribution is made by molecules for which $\theta \lesssim \tau$, where τ is the average time of flight of a molecule across the resonant cavity. Hence we are interested in the average value of $\hat{\rho}$ obtained as follows:

$$\hat{\rho}(t) = \int_0^\infty \hat{\rho}(t, t_0 = t - \theta) f(\theta)\, d\theta, \tag{5}$$

where $f(\theta)$ is a function that has been normalized to unity and decreases rapidly enough for $\theta > \tau$. The problem is simplified considerably if $f(\theta)$ is taken in the form

$$f(\theta) = (1/\tau)\, e^{-\theta/\tau}. \tag{6}$$

Such an approximation was used by Basov and Prokhorov* (Ref. 1).

It is easy to show by use of (2), (5), and (6) that $\hat{\rho}$ (t) satisfies the equation

$$\partial\hat{\rho}/\partial t = (-i/\hbar)(H\hat{\rho} - \hat{\rho}H) - \tau^{-1}(\hat{\rho} - \hat{\rho}_0). \tag{7}$$

Here $\hat{\rho}_0 \equiv \hat{\rho}(t, t_0 = t)$ is the density matrix at the instant of entry of the molecules into the resonator.

Now we assume that the molecule has only two levels E_1 and E_2 ($E_2 > E_1$). Then $\hat{\rho}$ is a two-rowed matrix. Further we assume that at the instant of entry into the resonator all molecules are in the state E_2. Then

$$(\rho_0)_{11} = (\rho_0)_{12} = (\rho_0)_{21} = 0;\ (\rho_0)_{22} = N_0, \tag{8}$$

where N_0 is the number of active molecules per unit volume in the absence of the field.

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*Strictly speaking, in a molecular beam (which is used in the molecular generator) P is a function of z and v

$$P = P(t;\, t - z/v),$$

where the z axis is along the beam, and v is the velocity of molecules in the beam. When we replace the resonator by the equivalent circuit, it is necessary to average P along z as follows [cf., e.g., Ref. 4],

$$P(t, v) = \int P(t,\, t - z/v)\, E_a(z)\, dz,$$

where $E_a(z)$ is the normal oscillation of the cavity. Next, this formula should be averaged over v. However, even the monochromatic approximation leads to rather cumbersome equations.

In the representation, in which H_0 is diagonal and has eigenvalues E_1 and E_2, Eq. (7) can be written in the form

$$\dot{\rho}_{12} + i(\omega_{12} + \tau^{-1})\rho_{12} = \frac{i}{\hbar}(\rho_{22} - \rho_{11})E\mu_{12}, \quad \dot{\rho}_{21} + (i\omega_{21} + \tau^{-1})\rho_{21} = -\frac{i}{\hbar}(\rho_{22} - \rho_{11})E\mu_{21},$$

$$\dot{\rho}_{11} = \frac{i}{\hbar}E(\mu_{12}\rho_{21} - \rho_{12}\mu_{21}) - \tau^{-1}\rho_{11}, \quad \dot{\rho}_{22} = \frac{i}{\hbar}E(\mu_{21}\rho_{12} - \rho_{21}\mu_{12}) - \tau^{-1}(\rho_{22} - N_0). \tag{9}$$

From (4) we find (assuming that $\mu_{11} = \mu_{22} = 0$)

$$P = \rho_{12}\mu_{21} + \rho_{21}\mu_{12}. \tag{4a}$$

From (9) and (4a) it is not difficult to derive (taking into account that $\omega_{21} = -\omega_{12} \equiv \omega_2 > 0$)

$$\ddot{P} + (2/\tau)\dot{P} + (\omega_2^2 + \tau^{-2})P = -2NE\,|\mu_{12}|^2\omega_2/\hbar, \quad \dot{N} + \tau^{-1}N - (2/\hbar\omega_2)E(\dot{P} + \tau^{-1}P) = \tau^{-1}N_0. \tag{10}$$

These are the desired equations connecting E and P; hence in these equations N stands for the number of active molecules in the presence of the field E: $N = \rho_{22} - \rho_{11}$ (for E = 0 and $N = N_0$).

If we set $E = E_0 \cos \omega t$, then it is easily shown from Eqs. (10) that under neglect of higher harmonics and for ω close to ω_2 an expression for the complex dielectric constant is obtained that takes into account the saturation effect and agrees with the expression derived in Ref. 1.

Thus a complete system of equations, describing the oscillation of the molecular generator, is given by Eqs. (1) and (10).

In conclusion I wish to thank Prof. V. L. Ginzburg for a thorough discussion of the results.

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 4 APRIL, 1958

*THE INTERACTION OF SHOCK WAVES WITH SMALL PERTURBATIONS. I**

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The interaction of oblique shock waves of arbitrary intensity with small perturbations is considered in the linear approximation, within the framework of a two-dimensional stationary problem. The flow behind the shock wave is assumed to be supersonic. The general results are applied to the study of the interaction with weak discontinuities, weak shock waves and with the interface between two media of slightly different acoustical properties.

1. STATEMENT OF THE PROBLEM

In our note of Ref. 1, a complete classification was given of the cases of the interaction of shock waves with weak discontinuities. All these cases can be considered quantitatively by means of perturbation theory. Each weak discontinuity can be represented as a small perturbation superimposed on a smooth distribution of hydrodynamic quantities. If we are interested in the behavior of all quantities in the neighborhood of the intersection of the weak discontinuity with the shock, then the flow in the vicinity of the point of interaction (which is not perturbed by the weak discontinuity) can be considered homogeneous and the shock wave can be regarded as plane. The incident weak discontinuity can be thought of as a small "incoming" perturbation, as a result of the interaction of which with the shock wave, small "outgoing" perturbations are created. The smallness of the perturbation permits us to expand the hydrodynamical equations and the boundary conditions on the shock wave in a series in the perturbations; limiting ourselves to linear terms, we can obtain the complete solution of interest to us.

Thus, let us consider a plane shock wave in homogeneous flow, and, given the incoming perturbation, let us find the outgoing perturbation. We choose the y axis in a direction perpendicular to the shock wave, and let the x axis lie in the plane of the shock wave, with its direction coinciding with the direction of the x component of the velocity; let matter flow into the shock from below.

We denote by p_0, v_0, V_0, s_0, the pressure, velocity, specific volume and entropy of the unperturbed flow before the shock wave (for $y < 0$), and by p, v, V, s, the corresponding quantities after the wave (for $y > 0$); we shall denote all small perturbations of the corresponding quantities by the symbol δ. Expanding the hydrodynamical equations in a series in small values of the perturbation, and limiting ourselves to linear terms, we obtain a system of linear equations with constant coefficients for the flow behind the shock wave:

$$(\mathbf{v}\nabla)\,\delta\mathbf{v} + V\nabla\delta p = 0, \tag{1.1}$$

$$\operatorname{div}\delta\mathbf{v} + (V/c^2)\,\mathbf{v}\nabla\delta p = 0, \tag{1.2}$$

$$(\mathbf{v}\nabla)\,\delta s = 0. \tag{1.3}$$

Similar equations are obtained for the flow in front of the shock if all quantities are given the index 0.

If the flow behind the shock is supersonic, the equations (1.1) — (1.3) are everywhere hyperbolic and have a general solution which is easily found by the method of characteristics. The latter, as is well known, consists of finding linear characteristic combinations of the desired functions which are constant along the determined lines (in our case, straight lines); these characteristics form a one-parameter set. We first note that Eq. (1.3), which has the general solution.

$$\delta s = f(x - y/k_0), \tag{1.4}$$

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*The two papers published here belonged to S. P. D'iakov, who perished tragically in 1954. They were prepared for publication by his colleagues from his manuscripts — L. Landau.

†Deceased.

where $k_0 = v_y/v_x$ and f is an arbitrary function of its argument, is already in characteristic form. The solution of (1.4) shows that δs is a characteristic quantity, which is constant along each of the flow lines, $y - k_0x = C$, which form a one-parameter set of characteristics. Graphically speaking, the disturbance δs is "carried along" the line of flow.

To find the three other characteristic quantities,[2] we construct a linear combination of Eqs. (1.1) and (1.2) with coefficients α, β, γ:

$$(\alpha v_x+\gamma)\frac{\partial \delta v_x}{\partial x}+\alpha v_y\frac{\partial \delta v_x}{\partial y}+\beta v_x\frac{\partial \delta v_y}{\partial x}+(\beta v_y+\gamma)\frac{\partial \delta v_y}{\partial y}+\left(\alpha+\frac{v_x\gamma}{c^2}\right)V\frac{\partial \delta p}{\partial x}+\left(\beta+\frac{v_y\gamma}{c^2}\right)V\frac{\partial \delta p}{\partial y}=0,$$

and require that all the quantities δv_x, δv_y, δp in this equation be differentiated along the same direction i.e., that the ratio of the coefficient for the derivative with respect to x to the coefficient for the derivative with respect to y be the same for all quantities δv_x, δv_y, δp:

$$\frac{\alpha v_y}{\alpha v_x+\gamma}=\frac{\beta v_y+\gamma}{\beta v_x}=\frac{\beta+\gamma v_y/c^2}{\alpha+\gamma v_x/c^2}=k. \tag{1.5}$$

This condition means that the derivative of the combination

$$I=\alpha v_y \delta v_x+(\beta v_y+\gamma)\,\delta v_y+(\beta+v_y\gamma/c^2)\,V\delta p$$

along the direction with angular coefficient k is equal to zero, i.e., this combination is characteristic for the set of characteristics $y - kx = C$. In other words, the quantity I is "carried along" the characteristic $y - kx = C$.

We can write the condition in the form of a system of linear homogeneous equations in α, β, γ:

$$(v_y-v_xk)\,\alpha-k\gamma=0,\ \ (v_y-v_xk)\,\beta+\gamma=0,\ \ -k\alpha+\beta+c^{-2}(v_y-v_xk)\,\gamma=0. \tag{1.6}$$

The system has solutions differing from zero if its determinant vanishes:

$$(v_y-v_xk)\,[(v_y-v_xk)^2-(1+k^2)\,c^2]=0. \tag{1.7}$$

The cubic equation (1.7) has three roots (real, if the motion is supersonic), each of which determines a set of rectilinear characteristics. To each root there corresponds a definite solution of the system (1.6) of α, β, γ, i.e., a definite linear combination I. Let us write out all these characteristic combinations.

1. $$k_0=v_y/v_x;\ \ \alpha=\beta v_x/v_y;\ \ \gamma=0. \tag{1.8}$$

The characteristic combination is

$$\mathbf{v}\,\delta\mathbf{v}+V\delta p=I_0\,(x-y/k_0).$$

It is obvious that the flow line $k_0 = v_y/v_x$ is a "twofold characteristic," since two characteristic quantities δs and I_0 are carried along it.

2. $$k_{\pm}=\frac{v_xv_y\pm c^2\sqrt{M^2-1}}{v_x^2-c^2};\ \ \alpha=-k\beta;\ \ \gamma=-(v_y-v_xk)\,\beta. \tag{1.9}$$

The characteristic combinations are

$$v_x\delta v_y-v_y\delta v_x\pm\sqrt{M^2-1}\,V\delta p=I_{\pm}\,(x-y/k),$$

where $M = v/c$. These combinations are perturbations of the acoustic type, which are propagated along the corresponding characteristics.

Assuming that all the characteristic combinations vanish save one, we obtain four different independent solutions of the hydrodynamical equations: each of these solutions contains only a single arbitrary function. It is appropriate to take four equations for the fundamental independent solutions: we can represent an arbitrary solution in the form of their linear combinations. Let us now obtain these fundamental solutions.

1. Setting $I_0 = I_\pm = 0$; $\delta s = \delta s\,(y - k_0x)$, we shall have, in correspondence with (1.7) and (1.8), a solution which describes the propagation of the entropy perturbation, for which

$$\delta v=0;\ \ \delta p=0;\ \ \delta s=\delta s\,(y-k_0x);\ \ \delta V=(\partial V/\partial s)_p\,\delta s. \tag{1.10}$$

2. Assuming $\delta s = 0$, $I_\pm = 0$, we get a solution which corresponds to the propagation of a perturbation of the curve of velocity

$$\delta p = 0;\; v_x \delta v_y - v_y \delta v_x = 0;\; \mathbf{v}\,\delta\,\mathbf{v} = I\,(y - k_0 x). \tag{1.11}$$

Both these solutions, which are connected with the lines of flow, will for convenience be joined in a single "entropy-vortical" solution.

3. Assuming $\delta s = 0$, $I_0 = I_+ = 0$, we get a solution of the acoustic perturbation type:

$$\delta p = \delta p\,(y - k_{-}x),$$
$$\delta v_x = \frac{-v_x + v_y\sqrt{M^2-1}}{M^2c^2}\,V\delta p,\; \delta v_y = -\frac{v_x\sqrt{M^2-1} + v_y}{M^2c^2}\,V\delta p. \tag{1.12}$$

4. Finally, for $\delta s = 0$, $I_0 = I_- = 0$, we get an acoustic perturbation with reverse direction

$$\delta p = \delta p\,(y - k_{+}x);$$
$$\delta v_x = -\frac{v_y\sqrt{M^2-1} + v_x}{M^2c^2}\,V\delta p;\; \delta v_y = \frac{v_x\sqrt{M^2-1} - v_y}{M^2c^2}\,V\delta p. \tag{1.13}$$

The position of the characteristics relative to the shock is shown in Fig. 1. The waves (points) and the trajectories (solid lines) are shown in the middle of Fig. 1 for the obstacles A and A′, which are found in front of the shock (A) and behind the shock (A′). These waves and trajectories represent the characteristics of the equations of gas dynamics. Inasmuch as they describe the propagation of a disturbance produced at a point, we can describe the determined directions of propagation — from the points A or A′ — by means of the characteristics. (See Fig. 1, where the directions are shown by arrows.)

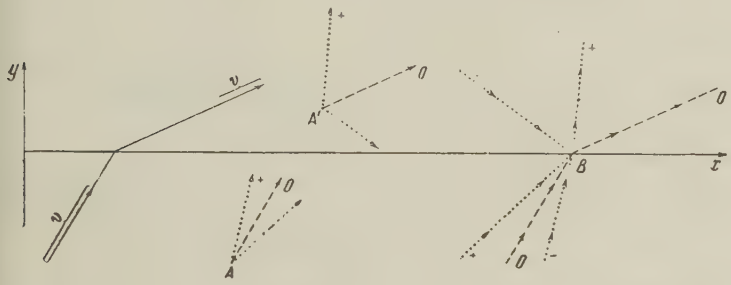

FIG. 1. ——— -shock wave — — — characteristics of the entropy-vortical perturbation, coinciding with the lines of flow, ··· — characteristics of the acoustic perturbation.

The relations between v_y and c for $y < 0$ and $y > 0$, which must hold for the undisturbed shock wave, inevitably lead to the result that for $y < 0$, all three characteristics, originating from an arbitrary point A, abut against the surface of the shock wave; for $y > 0$, one and only one of the characteristics which emanate from A′ abuts the surface of the wave. Keeping the deflections and the directions of propagation of the characteristics, we shall transform them so that the point of intersection coincides with the wave front B; we thus survey all the characteristics which converge on the given point B and emerge from it, and obtain the picture shown on the right side of Fig. 1.

All the drawings are made for a gas with $\gamma = 1.4$, for shock waves in which the pressure increases 10 times and the horizontal velocity exceeds by 1.7 times the sound velocity before the compresion; in this case, we obtain $c/v = 0.87$ (for $y > 0$) in back of the compression.

In order to understand the distribution of the characteristics graphically, let us first represent the unperturbed horizontal surface of the shock and the flow normal to it. As is well known, in such a current, $v_y > c$ for $y < 0$ and $v_y < c$ for $y > 0$; the material velocity relative to the wave is supersonic before the front and subsonic behind the front of the shock. Let us now set up the point of view of an observer moving with velocity v_x with the front: the surface of the shock then remains at rest, and we get for the velocity of the gas the horizontal component v_x both in front of and behind the front. The velocity and trajectory vectors that are shown on the left in Fig. 1. We choose v_x sufficiently large that the motion is supersonic even behind the front, $v = \sqrt{v_x^2 + v_y^2} > c$ for $y > 0$.

An obstacle placed in supersonic flow gives two stationary Mach waves, making equal angles with the trajectory of the particles of the gas which abut the obstacle. The sine of the angle is equal to the ratio of the sound velocity to the flow velocity.

On the side $y < 0$, a disturbance of any of the types enumerated can be incident on the shock wave.

From the side $y > 0$, only an accoustic disturbance of type 3 (sound disturbance "catches up" with the shock wave) can occur. Thus interaction of the shock is possible with five types of disturbances. As the result of the interaction, disturbances are formed of types 1, 2, 4, which emanate from the wave in the direction $y > 0$ along the characteristics k_0 and k_+. For a given type of incident disturbance, the solution of the problem will be a combination of the incident and (in the general case) three emergent disturbances. This linear combination is determined from the boundary conditions on the shock wave, into the derivation of which we shall now go.

Let the equation of the excited shock wave be $y = \eta(x)$, where η is a small quantity; the components of the unit vectors in the directions concerned and normal to the wave (in the approximation of interest to us) are $\mathbf{t}(1, \eta')$ and $\mathbf{n}(-\eta', 1)$, respectively.

Expanding the continuity condition of the tangential component of the velocity

$$(\mathbf{v}_0 + \delta\,\mathbf{v}_0)\,\mathbf{t} = (\mathbf{v} + \delta\,\mathbf{v})\,\mathbf{t}$$

in terms of a small disturbance, and limiting ourselves to linear terms, we find

$$\delta v_{0x} - \delta v_x = (v_y - v_{0y})\,\eta'. \tag{1.14}$$

In a similar manner, the condition which connects the velocity jump with the pressure jump

$$(\mathbf{v}_0 + \delta\,\mathbf{v}_0)\,\mathbf{n} - (\mathbf{v} + \delta\,\mathbf{v})\,\mathbf{n} = [(p - p_0 + \delta p - \delta p_0)\,(V_0 - V + \delta V_0 - \delta V)]^{1/2}$$

gives, after expansion,

$$\frac{\delta v_{0y} - \delta v_y}{v_{0y} - v_y} = \frac{1}{2}\left[\frac{\delta p - \delta p_0}{p - p_0} + \frac{\delta V_0 - \delta V}{V_0 - V}\right]. \tag{1.15}$$

The equation for the Hugoniot adiabatics can be written in the general form $V = V(p, p_0, V_0)$, whence

$$\delta V = \frac{\partial V}{\partial p}\,\delta p + \frac{\partial V}{\partial p_0}\,\delta p_0 + \frac{\partial V}{\partial V_0}\,\delta V_0. \tag{1.16}$$

Finally, there remains the condition which determines the matter velocity in front in terms of the jump of the quantities in the discontinuity:

$$[(\mathbf{v}_0 + \delta\,\mathbf{v}_0)\,\mathbf{n}]^2 = (V_0 + \delta V_0)^2\,(p - p_0 + \delta p - \delta p_0)/(V_0 - V + \delta V_0 - \delta V),$$

which, after expansion, yields

$$\frac{2}{v_{0y}}\,(\delta v_{0y} - v_{0x}\eta') = \frac{2\delta V_0}{V_0} + \frac{\delta p - \delta p_0}{p - p_0} - \frac{\delta V_0 - \delta V}{V_0 - V}. \tag{1.17}$$

The system of boundary conditions (1.14) — (1.16) allows us to determine the solution uniquely.

2. SOLUTIONS OF THE EQUATION

We proceed to find the solution of interest to us, in the case in which the external disturbance takes place from below ($y < 0$). Assuming that all the quantities which describe the onset of the distrubance are given on the shock wave as a function of x, we determine all the quantities on the shock wave (for $y = 0$) which describe the onset of the disturbance, dividing them into parts corresponding to the independent solutions described. Eliminating $\delta V'$ and η' from (1.15) and (1.17), by means of (1.14) and (1.16), we get the following relations:

$$\begin{gathered}
\frac{\delta v_{0y} - \delta v_y}{v_{0y} - v_y} = \frac{1}{2}\left[\frac{\delta p}{p - p_0}\left(1 - j^2\frac{\partial V}{\partial p}\right) - \frac{\delta p_0}{p - p_0}\left(1 + j^2\frac{\partial V}{\partial p_0}\right) + \frac{\delta V_0}{V_0 - V}\left(1 - \frac{\partial V}{\partial V_0}\right)\right], \\
\frac{\delta v_{0x} - \delta v_x}{v_{0y} - v_y} = -\frac{\delta v_{0y}}{v_{0x}} + \frac{v_{0y}}{2v_{0x}}\left[\frac{2\delta V_0}{V_0} + \frac{\delta p}{p - p_0}\left(1 + j^2\frac{\partial V}{\partial p}\right)\right. \\
\left. - \frac{\delta p_0}{p - p_0}\left(1 - j^2\frac{\partial V}{\partial p_0}\right) - \frac{\delta V_0}{V_0 - V}\left(1 - \frac{\partial V}{\partial V_0}\right)\right].
\end{gathered} \tag{2.1}$$

Here by $\delta\mathbf{v}$ we mean the linear combination of two quantities, one of which is determined by the entropy vortex $[\delta\,\mathbf{v}^{(1)}]$ solution, the other, by the departing sound $[\delta\,\mathbf{v}^{(2)}\colon \delta\,\mathbf{v} = \delta\,\mathbf{v}^{(1)} + \delta\,\mathbf{v}^{(2)}]$. We multiply the first of these equations by v_x, the second by v_y, and subtract one from the other:

$$2\cot\varphi\,(v_y\delta v_x^{(2)} - v_x\delta v_y^{(2)}) - \left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial \rho}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\delta p = F(x), \tag{2.2}$$

where

$$F(x) = 2\cot\varphi\,(v_y\delta v_{0x} - v_x\delta v_{0y}) - \left[V_0\cot^2\varphi\left(1 + j^2\frac{\partial V}{\partial p_0}\right) - V\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0 + 2j\frac{V}{V_0}(V_0 - V)\,\delta v_{0y}$$
$$+ j^2\left[\left(1 - \frac{\partial V}{\partial V_0}\right)(V + V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0 - V)\right]\delta V_0. \tag{2.3}$$

In the derivation, we made use of the relations

$$v_x = v_{0x} = jV_0\cot\varphi; \quad v_y = jV; \quad v_{0y} = jV_0.$$

The contribution of the entropy-vertical solution on the left hand side (2.2) is equal to zero identically, by virtue of (1.11).

Furthermore, according to (1.13),

$$v_y\delta v_x^{(2)} - v_x\delta v_y^{(2)} = -\sqrt{M^2 - 1}\,V\delta p.$$

Substituting this in (2.2), we find the expression for δp on the shock:

$$\delta p(x) = -F(x)/D_+, \tag{2.4}$$

$$D_+ = V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right) + 2\cot\varphi\sqrt{M^2 - 1}\,V. \tag{2.5}$$

Further from, (1.13).

$$\delta v_x^{(2)}(x) = (v_y\sqrt{M^2 - 1} + v_x)\,VF(x)/M^2c^2D_+, \qquad \delta v_y^{(2)}(x) = (v_y - v_x\sqrt{M^2 - 1})\,VF(x)/M^2c^2D_+. \tag{2.6}$$

We now note that $\delta V = \delta V^{(1)} + \delta V^{(2)}$, where $\delta V^{(1)} = (\partial V/\partial s)_p\,\delta s$ is the contribution of the entropy solution and $\delta V^{(2)} = -(V/c)^2\delta p$ that of the sound. Substituting this in (in (1.16), we find:

$$\delta V^{(1)}(x) = \left[\frac{\partial V}{\partial p} + \frac{V^2}{c^2}\right]\delta p(x) + \frac{\partial V}{\partial p_0}\delta p_0(x) + \frac{\partial V}{\partial V_0}\delta V_0(x) = -\frac{\partial V/\partial p + V^2/c^2}{D_+}F(x) + \frac{\partial V}{\partial p_0}\delta p_0(x) + \frac{\partial V}{\partial V_0}\delta V_0(x); \tag{2.7}$$

$$\delta s(x) = (\partial s/\partial V)_p\,\delta V_1(x).$$

We multiply the first equation of (2.1) by v_y, and the second by v_x, and combine:

$$\mathbf{v}(\delta\mathbf{v}^{(1)} + \delta\mathbf{v}^{(2)}) = \mathbf{v}_0\delta\mathbf{v}_0 - \frac{1}{2}\delta p\left[V\left(1 - j^2\frac{\partial V}{\partial p}\right) + V_0\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]$$
$$+ \frac{1}{2}\delta p_0\left[V\left(1 + j^2\frac{\partial V}{\partial p_0}\right) + V_0\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right] - \frac{1}{2}j^2(V_0 - V)\left(1 + \frac{\partial V}{\partial V_0}\right)\delta V_0.$$

Noting that according to (1.13), $\mathbf{v}\delta\mathbf{v}^{(2)} = -V\delta p$, we finally obtain

$$I_0(x) = \mathbf{v}\,\delta\mathbf{v}^{(1)}(x) = \mathbf{v}_0\,\delta\mathbf{v}_0(x) + \frac{1}{2}\,\frac{(V_0 - V)\left(1 + j^2\dfrac{\partial V}{\partial p}\right)}{D_+}F(x)$$
$$+ \frac{1}{2}\left[V\left(1 + j^2\frac{\partial V}{\partial p_0}\right) + V_0\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0(x) - \frac{1}{2}j^2(V_0 - V)\left(1 + \frac{\partial V}{\partial V_0}\right)\delta V_0(x),$$
$$\delta v_x^{(1)}(x) = (v_x/M^2c^2)\,I_0(x); \quad \delta v_y^{(1)}(x) = (v_y/M^2c^2)\,I_0(x). \tag{2.8}$$

We obtained values of the quantities which describe the excitation reaching the shock wave as a function of x. Now, substituting $x - y/k_0$ for x in δV_1 and I_0, and $x - y/k_+$ for x in δp, and summing the acoustic and the entropy solutions, we find the total solution of the hydrodynamic equations for $y > 0$:

$$\delta p(x, y) = -F(x - y/k_+)/D_+, \quad \delta V(x, y) = \frac{V^2}{c^2D_+}F\left(x - \frac{y}{k_+}\right) - \frac{\partial V/\partial \rho + V^2/c^2}{D_+}F\left(x - \frac{y}{k_0}\right) + \frac{\partial V}{\partial p_0}\delta p_0\left(x - \frac{y}{k_0}\right) + \frac{\partial V}{\partial V_0}\delta V_0(x),$$
$$\delta v_x(x, y) = \frac{v_y\sqrt{M^2 - 1} + v_x}{M^2c^2D_+}VF\left(x - \frac{y}{k_+}\right) + \frac{v_x}{M^2c^2}I_0\left(x - \frac{y}{k_0}\right)$$
$$\delta v_y(x, y) = \frac{v_x - v_x\sqrt{M^2 - 1}}{M^2c^2D_+}VF\left(x - \frac{y}{k_+}\right) + \frac{v_y}{M^2c^2}I_0\left(x - \frac{y}{k_0}\right). \tag{2.9}$$

Equations (2.9) represent the local linear relation of the departing perturbations with the given incident one. It follows from these equations that the departing perturbations will be of the same type as the incoming. Since, if the arriving perturbation is a weak acoustic shock or the tangential discontinuity is of low intensity, the departing perturbations will also consist of a weak shock wave (or weak rarefaction wave), and a tangential discontinuity of low intensity. If the incident perturbation is a discontinuity of arbitrary hydrodynamical quantities (a weak or a weak tangential discontinuity), the weak and weak tangential discontinuities which are produced as the result of the interaction will also be discontinuities of arbitrary hydrodynamical quantities, etc.

We now determine the form of the perturbation of a shock wave. It is appropiate to characterize it by the curvature K, which is equal to $\eta''(x)$ in our approximation. Substituting the values of δp and δV on the shock in Eq. (1.17), we find

$$K(x) = \eta''(x) = \tan\varphi\left[\frac{\delta v'_{0y}(x)}{v_{0y}} + \frac{1}{2}\frac{\delta p'_0(x)}{p-p_0}\left(1 - j^2\frac{\partial V}{\partial p_0}\right) + \frac{2V - V_0\left(1+\frac{\partial V}{\partial V_0}\right)}{2V_0(V_0-V)}\delta V'_0(x) + \frac{1}{2}\frac{F'(x)}{D(p-p_0)}\left(1 + j^2\frac{\partial V}{\partial p}\right)\right. . \tag{2.10}$$

We see that the curvature of a shock wave is determined by the first derivatives of hydrodynamical quantities

3. INTERACTION OF THE SHOCK WAVE WITH A WEAK ACOUSTIC WAVE WHICH ENTERS FROM AN INCOMPRESSIBLE GAS

Let us consider the interaction with a weak acoustic shock wave which can fall along either of two characteristics k_{+0} and k_{-0}; in correspondence with this we must distinguish two cases (Fig. 2a and Fig. 2b). As a result of the interaction there arise, for $y > 0$, a weak shock wave (or a rarefaction wave of low intensity), which is associated with the characteristic k_+, and a tangential discontinuity of low intensity, which is propagated along the characteristic k_0. We compute the intensity of the outgoing discontinuities.

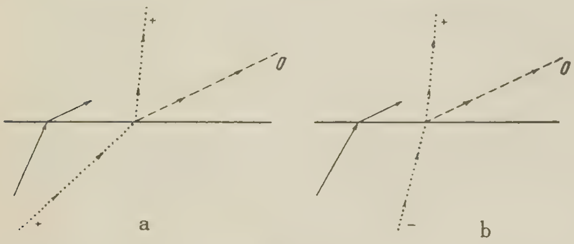

FIG. 2

We choose the origin of the coordinates as the point of intersection of the weak shock wave with the front of the strong shock wave. The incident weak shock is an excitation of the acoustic type:

$$\delta p_0(x, y) = \begin{cases} 0 & \text{for } x < y/k_{\pm 0} \\ \delta p_0 \equiv \text{const} & \text{for } x > y/k_{\pm 0}. \end{cases} \tag{3.1}$$

The quantity δp_0 can serve as a measure of the intensity of the wave. The quantities δv_0 and δV_0 are connected with δp_0 by Eqs. (1.12) and (1.13). Substituting these quantities in (2.3) and (2.4), we find the following expression for δp:

$$\delta p(x) = a^{(\pm)}_{pp_0}\delta p_0(x), \tag{3.2}$$

where

$$a^{(\pm)}_{pp_0} = \frac{1}{D_+}\left\{\frac{2j^2V_0}{M_0^2c_0^2}\left[\pm\cot\varphi\sqrt{M_0^2-1}\,(V^2+V_0^2\cot^2\varphi) - (V_0-V)(V_0\cot^2\varphi - V)\right] + \left[V_0\cot^2\varphi\left(1+j^2\frac{\partial V}{\partial p_0}\right) - V\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right]\right.$$
$$\left. + \frac{j^2V_0^2}{c_0^2}\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi) - \frac{2V}{V_0}(V_0-V)\right]\right\}$$

is the "transmission coefficient." If $a_{pp_0} > 0$, the outgoing acoustic perturbation is a weak shock wave; if $a_{pp_0} < 0$, it is a rarefaction wave of low intensity. We turn our attention to the fact that the quantity D can become zero; in this case, the "transmission coefficient" $a^{(\pm)}_{pp_0}$ tends to infinity — a singular "resonance" is observed — i.e., an amplification of the previous shock wave. It is easy to understand what the possibility of such a resonance means physically. For $D_+ = 0$, $a^{(\pm)}_{pp_0} = \infty$, a solution of the equation is possible with finite δp in the absence of a perturbation, which is incident on the shock wave in front; we are dealing with a spontaneous "splitting off" of a weak shock from the shock wave. Thus the condition $D = 0$ stands for such a connection of the parameters of the shock wave for which a Mach reflection of a weak shock wave is possible.[3,4]

Now let us compute the intensity of the tangential discontinuity. In accord with (2.7), we have

$$\delta V^{(1)}(x) = \left(\frac{\partial V}{\partial p} + \frac{V^2}{c^2}\right)\delta p(x) + \left(\frac{\partial V}{\partial p_0} - \frac{V_0^2}{c_0^2}\frac{\partial V}{\partial V_0}\right)\delta p_0(x) = a_{Vp_0}\delta p_0(x),$$
$$a_{Vp_0} = a_{pp_0}(\partial V/\partial p + V^2/c^2) + \partial V/\partial p_0 - (V_0/c_0)^2 \partial V/\partial V_0. \tag{3.3}$$

The jump of the tangential component of the velocity at a tangential discontinuity is determined by the quantity I_0. According to (2.8), we get

$$I(x) = a_{Ip_0}\delta p_0, \quad a_{Ip_0} = -\tfrac{1}{2}(V_0 -, V)\left[\left(1 + j^2\frac{\partial V}{\partial p}\right)a_{pp_0} + \left(1 + j^2\frac{\partial V}{\partial p_0}\right) - \frac{j^2V_0^2}{c_0^2}\left(1 + \frac{\partial V}{\partial V_0}\right)\right]. \tag{3.4}$$

The shock at the origin will obviously have an inclination at a finite angle, proportional to the intensity of the incident weak shock wave. Making use of (2.10), we get the following formula for the angle of inclination χ:

$$\chi \approx a_{\chi p_0}\delta p_0, \tag{3.5}$$

where

$$a_{\chi p_0} = \tan\varphi\left\{\frac{\pm\cot\varphi\sqrt{M_0^2-1}-1}{M_0^2c_0^2}V_0 + \frac{1}{2(p-p_0)}\left[1 - j^2\frac{\partial V}{\partial p_0} - a_p\left(1 + j^2\frac{\partial V}{\partial p}\right) - \frac{j^2V_0}{2c_0^2}\left[2V - V_0\left(1 + \frac{\partial V}{\partial V_0}\right)\right]\right]\right\}.$$

4. INTERACTION OF A WEAK DISCONTINUITY WITH A SHOCK WAVE

Now let us assume that a small perturbation with a weak discontinuity is incident on the shock wave; for definiteness, we shall consider that the weak discontinuity is a discontinuity of the derivatives of the hydrodynamical quantities. We choose the origin of the coordinates as the point of intersection of the weak discontinuity with the shock wave. Expansion of any quantity along the shock wave not far from the origin will have the form

$$f = \begin{cases} f(0) + (A_f - \Delta_f)x & \text{for } x < 0 \\ f(0) + A_f x & \text{for } x > 0, \end{cases} \tag{4.1}$$

where the quantity Δ_f is proportional in the proper fashion to the determined intensity of the incident weak discontinuity. The intensity of the weak discontinuity is appropriately determined as the jump of the derivative of the hydrodynamic quantity along the normal to the discontinuity bearing the characteristic $(\partial f/\partial n)_+ - (\partial f/\partial n)_-$; in this case we have agreed to consider as the positive direction of the normal the direction obtained from the direction of the characteristic turned through an angle of $\pi/2$ in the clockwise direction. Thus, the intensity of a weak discontinuity of the acoustic type will be determined by the jump in the pressure gradient along the normal $n_\pm$ to the acoustic characteristic $\Delta^{(\pm)} = (\partial\delta p/\partial n)_+ - (\partial\delta p/\partial n)_-$, the intensity of the weak tangential discontinuity — by the jumps of the entropy gradient and the quantity I_0:

$$\Delta^{(01)} = \left(\frac{\partial\delta s}{\partial n_0}\right)_+ - \left(\frac{\partial\delta s}{\partial n_0}\right)_-, \quad \Delta^{(02)} = \left(\frac{\partial\delta I_0}{\partial n_0}\right)_+ - \left(\frac{\partial\delta I_0}{\partial n_0}\right)_-$$

along the normal n_0 to the flow line bearing the discontinuity. The connection of quantities of the type Δ_f with Δ, $\Delta^{(1)}$, $\Delta^{(2)}$ is determined from elementary geometrical considerations.

Thus, it is easy to see that

$$\Delta_p = \frac{|k_{(\pm)}|}{\sqrt{1+k_\pm^2}}\Delta^{(\pm)} = \left|\frac{v_xv_y \pm c^2\sqrt{M^2-1}}{c(\pm v_x\sqrt{M^2-1}+v_y)}\right|\Delta^{(\pm)}, \quad \Delta_V = \frac{k_0}{\sqrt{1+k_0^2}}\left(\frac{\partial V}{\partial s}\right)_p\Delta^{(01)} = \frac{v_y}{Mc}\left(\frac{\partial V}{\partial s}\right)_p\Delta^{(01)},$$
$$\Delta_{I_0} = \frac{k_0}{\sqrt{1+k_0^2}}\Delta^{(02)} = \frac{v_y}{Mc}\Delta^{(02)}. \tag{4.2}$$

After these preliminary remarks, we consider the interaction of a weak discontinuity of the acoustic type with the shock wave. It is obvious that the mutual relations of the quantities Δ_f on the shock wave is the same as for the f themselves; therefore,

$$\Delta_p = a_{pp_0}\Delta_{p_0}, \quad \Delta_V = a_{Vp_0}\Delta_{p_0}, \quad \Delta_{I_0} = a_{Ip_0}\Delta_{p_0}, \tag{4.3}$$

where a_p, a_V, a_I are coefficients determined in Sec. 2.

Now, expressing Δ_{p_0}, Δ_p, Δ_V; Δ_I in terms of the intensity of the incident ($\Delta_0^{(\pm)}$) and the transmitted ($\Delta^{(+)}$, $\Delta^{(01)}$, $\Delta^{(02)}$), weak discontinuities in accord with (4.2), we obtain equations which connect the $\Delta^{(+)}$, $\Delta^{(01)}$, $\Delta^{(02)}$, with $\Delta_0^{(\pm)}$:

$$\Delta^{(+)} = a_{pp_0}^{(\pm)} \frac{c\,(v_x\sqrt{M^2-1}+v_y)}{v_xv_y+c^2\sqrt{M^2-1}} \left|\frac{v_{0x}v_{0y}\pm c_0^2\sqrt{M_0^2-1}}{c_0\,(\pm v_{0x}\sqrt{M_0^2-1}+v_{0y})}\right| \Delta_0^{(\pm)}, \quad \Delta^{(01)} = a_{Vp_0}\frac{Mc}{v_y}\left(\frac{\partial s}{\partial V}\right)_p \left|\frac{v_{0x}v_{0y}\pm c_0^2\sqrt{M_0^2-1}}{c_0\,(\pm v_{0x}\sqrt{M_0^2-1}+v_{0y})}\right| \Delta_0^{(\pm)},$$

$$\Delta^{(02)} = a_{Ip_0}\frac{Mc}{v_y}\left|\frac{v_{0x}v_{0y}\pm c_0^2\sqrt{M_0^2-1}}{c_0\,(\pm v_{0x}\sqrt{M_0^2-1}+v_{0y})}\right| \Delta_0^{(\pm)}. \tag{4.4}$$

We note that, close to "resonance" ($D_+ = 0$), the intensity of the transmitted discontinuities is strongly increased.

Let us determine the form of the shock wave close to the origin. The relation (2.10) shows that η' is continuous, but the curvature $K \approx \eta''(x)$ has a finite jump Δ_K at the origin, equal to

$$\Delta_K = a_{Kp_0}\left|\frac{v_{0x}v_{0y}\pm c_0^2\sqrt{M_0^2-1}}{c_0\,(\pm v_{0x}\sqrt{M_0^2-1}+v_{0y})}\right| \Delta_0^{(\pm)}. \tag{4.5}$$

5. INTERACTION WITH A SHOCK WAVE OF A TANGENTIAL DISCONTINUITY OF LOW INTENSITY AND OF AN ENTROPY PERTURBATION IN AN INCOMPRESSIBLE GAS

Let a tangential discontinuity of low intensity, which is characterized by two quantities δV_0 and I_0

$$\delta V_0(x, y) = \begin{cases} 0 & \text{for } x < y/k_0^{(0)} \\ \delta V_0 = \text{const} & \text{for } x > y/k_0^{(0)} \end{cases}$$
$$I_0(x, y) = \begin{cases} 0 & \text{for } x < y/k_0^{(0)} \\ I_0 = \text{const} & \text{for } x > y/k_0^{(0)}. \end{cases} \tag{5.1}$$

be incident along the characteristic $k_0^{(0)}$.

As a result of the interaction, there arise for $y > 0$ a tangential discontinuity of low intensity and a weak shock wave (or a rarefaction wave of low intensity) (Fig. 3).

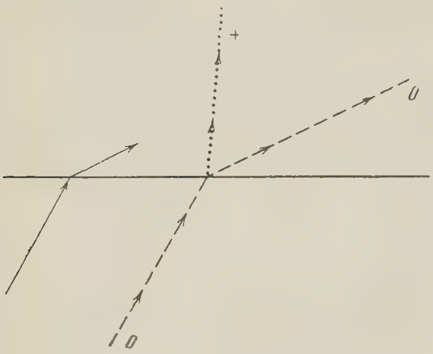

FIG. 3

Elementary calculations, by means of Eqs. (2.3) and (2.4), give the following expression for intensity of the transmitted shock wave (rarefaction wave)

$$\delta p = a_{pI_0}I_0 + a_{pV_0}\delta V_0, \qquad a_{pI_0} = \frac{2j^2(V_0-V)}{M_0^2c_0^2D_+}(V_0^2\cot^2\varphi - V)\,a_{pV_0}$$
$$= \frac{j^2}{D_+}\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi) - 2\,\frac{V}{V_0}(V_0-V)\right]\delta V_0. \tag{5.2}$$

The intensity of the transmitted tangential discontinuity is determined by means of Eqs. (2.7) and (2.8):

$$\delta V^{(1)} = a_{VI_0}I_0 + a_{VV_0}\delta V_0, \quad a_{VI_0} = \frac{2j^2(V_0-V)}{M_0^2c_0^2D_+}(V_0\cot^2\varphi - V)\left(\frac{\partial V}{\partial p}+\frac{V^2}{c^2}\right)I_0,$$
$$a_{VV_0} = \left\{\frac{\partial V}{\partial V_0} - \left(\frac{\partial V}{\partial p}+\frac{V^2}{c^2}\right)\frac{j^2}{D_+}\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0-V)\right]\right\}\delta V_0, \tag{5.3}$$

$$I = a_{II_0}I_0 + a_{IV_0}\delta V_0, \quad a_{II_0} = \left[1 - \frac{1}{2}\,\frac{j^2(V_0-V)^2}{M_0^2c_0^2D_+}(V_0\cot^2\varphi - V)\left(1+j^2\frac{\partial V}{\partial p}\right)\right], \tag{5.4}$$

$$a_{IV_0} = \frac{1}{2}\,j^2(V_0-V)\left\{-\left(1+\frac{\partial V}{\partial V_0}\right)+\frac{1}{D_+}\left(1+j^2\frac{\partial V}{\partial p}\right)\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0-V)\right]\right\}\delta V_0.$$

The shock wave at the point of intersection will make a break at a finite angle χ, which is determined from (2.10):

$$\chi = a_{\chi I_0} I_0 + a_{\chi V_0}\delta V_0, \qquad a_{\chi I_0} = \tan\varphi\left[1 - \frac{1}{D_+}\left(1 + j^2\frac{\partial V}{\partial p}\right)(V_0\cot^2\varphi - V)\right]\frac{I_0}{M_0^2 c_0^2},$$

$$a_{\chi V_0} = \frac{\tan\varphi}{2(V_0 - V)}\left\{\frac{V}{V_0} - 1 - \frac{\partial V}{\partial V_0} - \frac{1}{D_+}\left(1 + j^2\frac{\partial V}{\partial p}\right)\left[\left(1 - \frac{\partial V}{\partial V_0}\right)(V + V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0 - V)\right]\right\}. \tag{5.5}$$

We now consider the interaction of a weak tangential discontinuity (the type of discontinuity of derivatives of hydrodynamical quantities) with a shock wave, as a result of which weak and weak-tangential discontinuities are formed for $y > 0$.

Denoting the intensities of the incident weak tangential discontinuity by $\Delta_0^{(01)}$ and $\Delta_0^{(02)}$, we can easily find the expressions

$$\Delta^{(+)} = \frac{v_{0y}}{M_0 c_0}\frac{c(v_x\sqrt{M^2-1} + v_y)}{v_x v_y + c^2\sqrt{M^2-1}}\left[a_{pI_0}\Delta_0^{(02)} + a_{pV_0}\left(\frac{\partial V_0}{\partial s_0}\right)_{p_0}\Delta_0^{(01)}\right],$$

$$\Delta^{(01)} = \frac{V_0}{V}\frac{Mc}{M_0 c_0}\left(\frac{\partial s}{\partial V}\right)_p\left[a_{VI_0}\Delta_0^{(02)} + a_{pV_0}\left(\frac{\partial V_0}{\partial s_0}\right)_{p_0}\Delta_0^{(01)}\right], \qquad \Delta^{(02)} = \frac{V_0}{V}\frac{Mc}{M_0 c_0}\left[a_{II_0}\Delta_0^{(02)} + a_{IV_0}\left(\frac{\partial V_0}{\partial s_0}\right)_{p_0}\Delta_0^{(01)}\right]. \tag{5.6}$$

with the aid of (4.2).

The curvature of the shock wave at the origin will have a finite discontinuity Δ_K, equal to

$$\Delta_K = \frac{v_{0y}}{M_0 c_0}\left[a_{\chi V_0}\left(\frac{\partial V_0}{\partial s_0}\right)_{p_0}\Delta_0^{(01)} + a_{\chi I_0}\Delta_0^{(02)}\right]. \tag{5.7}$$

6. REFLECTION FROM A SHOCK OF AN ACOUSTIC WAVE PROPAGATED IN A COMPRESSIBLE GAS AND GENERATING AN ENTROPY-VORTICAL PERTURBATION IN SUCH A CASE

Let us now consider the case in which the incident perturbation comes from the region of flow behind the shock wave ("overtakes" the shock wave). The flow in front of the shock wave in this case is subsonic; in Eqs. (2.1), (2.2), we must set δV_0, δp_0, $\delta \mathbf{v}_0$ equal to zero. The perturbation behind the shock wave will be a combination of the incident acoustic perturbation and two outgoing perturbations — the sonic and the entropy-vortical:

$$\delta p = \delta p^{(0)} + \delta p^{(2)}, \quad \delta V = \delta V^{(0)} + \delta V^{(1)} + \delta V^{(2)}, \quad \delta \mathbf{v} = \delta \mathbf{v}^{(0)} + \delta \mathbf{v}^{(1)} + \delta \mathbf{v}^{(2)}, \tag{6.1}$$

where the quantities $\delta p^{(0)}$, $\delta V^{(0)}$, $\delta \mathbf{v}^{(0)}$ describe the incoming acoustic excitation.

Assuming $F = 0$ in Eq. (2.2) and substituting $\delta p^{(0)} + \delta p^{(2)}$ for δp, $\delta \mathbf{v}^{(2)} + \delta \mathbf{v}^{(0)}$ for $\delta \mathbf{v}^{(2)}$, we obtain the following ralations:

$$2\cot\varphi\,(v_y\delta v_x^{(0)} - v_x\delta v_y^{(0)}) - \left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\delta p^{(0)}$$
$$= -2\cot\varphi\,(v_y\delta v_x^{(2)} - v_x\delta v_y^{(2)}) + \left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\delta p^{(2)}. \tag{6.2}$$

Further, for the incoming perturbation, in accord with (1.12), we have

$$v_y\delta v_x^{(0)} - v_x\delta v_y^{(0)} = \sqrt{M^2-1}\,V\delta p^{(0)};$$

for the outgoing acoustic perturbation, in accord with (1.13),

$$v_y\delta v_x^{(2)} - v_x\delta v_y^{(2)} = -\sqrt{M^2-1}\,V\delta p^{(2)};$$

Substituting this in (6.2), we find the quantity $\delta p^{(2)}$ which characterizes the intensity of the outgoing acoustic perturbation

$$\delta p^{(2)} = -(D_-/D_+)\,\delta p^{(0)}, \tag{6.3}$$

where D_- differs from D_+ by the change in sign in the radical $\sqrt{M^2-1}$. The remaining quantities in the outgoing sound perturbation are computed in elementary fashion by means of Eq. (1.13).

We compute the entropy-vortical perturbation. With the aid of (1.15), we find

$$\delta V^{1)} = \frac{4}{D_+} \cot\varphi \sqrt{M^2-1}\, V \left(\frac{\partial V}{\partial p} + \frac{V^2}{c^2}\right) \delta p^{(0)}. \tag{6.4}$$

The quantity I_0, as is easy to see, is equal to

$$I_0 = -\frac{2}{D_+} \cot\varphi \sqrt{M^2-1}\, V (V_0 - V)\left(1 + j^2 \frac{\partial V}{\partial p}\right) \delta p^{(0)}. \tag{6.5}$$

The form of the shock wave is determined directly from (1.17):

$$\eta' = -\frac{2\sqrt{M^2-1}\, V}{j^2 (V_0 - V) D_+}\left(1 + j^2 \frac{\partial V}{\partial p}\right) \delta p^{(0)}. \tag{6.7}$$

Let the incident perturbation be a weak shock wave

$$\delta p^{(0)}(x, y) = \begin{cases} 0 & \text{for } x < y/k_- \\ \delta p^{(0)} = \text{const} & \text{for } x > y/k_-. \end{cases}$$

As a result of the interaction, a reflected weak shock wave (or a rarefaction wave of low intensity) and a tangential discontinuity of low intensity (Fig.4) are formed. In this case, Eq. (6.3) gives the intensity of the reflected shock wave (rarefaction wave) directly. Equations (6.4) and (6.5) determine the intensity of the tangential discontinuity.

The shock wave at the origin will undergo a break at a finite angle, the magnitude of which will be determined by the right side of Eq. (6.7).

If the incident excitation is a weak discontinuity (of the type of discontinuity of derivatives of hydrodynamical quantities), then, as a result of the interaction, outgoing weak and weak-tangential discontinuities are formed. Making use of the notations of the previous sections, we find the following relations between the intensities of the outgoing and the incoming weak discontinuities:

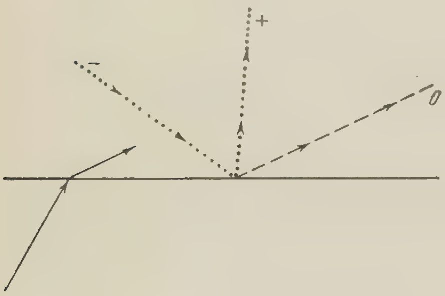

FIG. 4

$$\begin{aligned} \Delta &= -\frac{D_-}{D_+} \frac{|v_x v_y - c^2\sqrt{M^2-1}|}{v_x v_y + c^2 \sqrt{M^2-1}} \frac{v_x\sqrt{M^2-1} + v_y}{|v_x\sqrt{M^2-1} - v_y|}\Delta^{(0)}, \\ \Delta_1 &= +\frac{4\cot\varphi\sqrt{M^2-1}\, VM}{D_+}\left(\frac{\partial s}{\partial V}\right)_p \left(\frac{\partial V}{\partial p} + \frac{V^2}{c^2}\right)\left|\frac{v_x v_y - c^2\sqrt{M^2-1}}{v_y (v_x \sqrt{M^2-1} - v_y)}\right| \Delta^{(0)}, \\ \Delta_2 &= -\frac{2\cot\varphi\sqrt{M^2-1}\, VM}{D_+}(V_0 - V)\left(1 + j^2\frac{\partial V}{\partial p}\right)\left|\frac{v_x v_y - c^2\sqrt{M^2-1}}{v_y(v_x\sqrt{M^2-1} - v_y)}\right|\Delta^{(0)}. \end{aligned} \tag{6.8}$$

At the point of intersection with a weak discontinuity, the curvature of the shock wave has a finite discontinuity, equal to

$$\Delta_K = -\frac{2\sqrt{M^2-1}\, V}{j^2 (V_0 - V) D_+}\left(1 + j^2 \frac{\partial V}{\partial p}\right)\left|\frac{v_x v_y - c^2\sqrt{M^2-1}}{c(v_x\sqrt{M^2-1} - v_y)}\right|\Delta^{(0)}. \tag{6.9}$$

7. CONCLUSIONS

1. The interaction of a shock wave of arbitrary intensity with small perturbations has been considered in the case in which the motion behind the shock wave is supersonic.

2. The interaction of a shock wave of arbitrary intensity with a weak shock wave and a tangential discontinuity of low intensity has been considered, as a result of which interaction, an outgoing weak shock wave and a tangential discontinuity of low intensity are produced. Equations have been obtained connecting the intensity of the resultant discontinuities with the incident intensity.

3. The interactions of a shock wave with a weak and a weak-tangential discontinuity have been considered. As a result of these, outgoing weak and weak-tangential discontinuities are formed.

4. A singular phenomenon of "resonance" has been discovered, which leads to a significant amplification of the intensity of the incoming discontinuities. The connection is pointed out of this phenomenon with the possibility of splitting of the shock wave.

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INTERACTION OF SHOCK WAVES WITH SMALL PERTURBATIONS II.

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Consideration is given to the general case of the interaction of small perturbations with a shock wave of arbitrary intensity within the framework of a two dimensional stationary problem in the case of subsonic flow behind the shock wave. A solution is given for the interaction of weak and weak-tangential discontinuities with the shock wave. In both cases, the weak-tangential discontinuity formed behind the shock wave possesses a specific logarithmic discontinuity.

1. INTRODUCTION

In the previous paper,[1] the perturbation theory was given for the shock wave with supersonic flow behind it; with the aid of perturbation methods, problems were considered of the interaction of weak singularities with the shock wave. The purpose of the present work is the consideration of analogous interactions in the case of subsonic flow behind the shock wave.

For subsonic flow behind the shock wave, the equations of hydrodynamics are elliptic and the method of characteristics used in the previous paper are not applicable.

Let us consider a plane shock wave in one dimensional flow. As the y axis we choose the perpendicular to the wave in the direction of the normal component of the velocity. Let the x axis lie in the plane of the wave and be directed along the tangent to the component of the velocity.

Inasmuch as the motion is supersonic for $y < 0$, all the incident perturbations are described by solutions obtained in Sec. 1 of the previous paper. For $y > 0$, the system of hydrodynamic equations will possess only one set of real characteristics — flow lines along which the perturbation of the entropy and the curl of the velocity are "transferred." Therefore the entropy-vortical solution is preserved here also. Two sets of characteristics become imaginary. To find the solutions corresponding to the elliptic equation it is expedient to reduce it to Laplace's equation.

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The linearized hydrodynamic equations have the form

$$(\mathbf{v}\nabla)\,\delta\mathbf{v} + V\nabla\delta p = 0, \tag{1.1}$$

$$\operatorname{div}\delta\mathbf{v} + (V/c^2)\,\mathbf{v}\nabla\delta p = 0, \tag{1.2}$$

$$(\mathbf{v}\nabla)\,\delta s = 0. \tag{1.3}$$

Taking the divergence of (1.1) and eliminating div $\delta\mathbf{v}$ with the help of (1.2), we obtain the following elliptic equation for δp:

$$\Delta\delta p - c^{-2}(\mathbf{v}\nabla)^2\,\delta p = 0. \tag{1.4}$$

For the reduction of this equation to Laplace's equation, we undertake a transformation to a new variable by the substitutions

$$\sqrt{1-M^2}\,\xi = (1 - v_y^2/c^2)\,x + (v_x v_y/c^2)\,y, \quad \eta = y, \tag{1.5}$$

after which, as is easily shown, Eq. (1.4) takes the form

$$\partial^2\delta p/\partial\xi^2 + \partial^2\delta p/\partial\eta^2 = 0. \tag{1.6}$$

Thus δp is a harmonic function of the variables ξ, η.

It is appropriate to subject the independent solution of the hydrodynamic equations with $\delta p \neq 0$ to the conditions

$$\delta s = 0, \; I_0 = \mathbf{v}\delta\mathbf{v} + V\delta p = 0 \tag{1.7}$$

similarly to what was done in the previous paper. Taking the gradient of the latter condition

$$\nabla(\mathbf{v}\delta\mathbf{v}) + V\nabla\delta p \equiv (\mathbf{v}\nabla)\,\delta\mathbf{v} + [\mathbf{v}\operatorname{curl}\delta\mathbf{v}] + V\nabla\delta p = 0$$

and comparing with (1.1), we can establish the fact that

$$\operatorname{curl}\delta\mathbf{v} = 0, \quad \text{i.e.} \quad \delta\mathbf{v} = \nabla\varphi, \tag{1.8}$$

where φ is the velocity potential. It is easy to show that φ satisfies the equation (1.4), i.e., it is a harmonic function of the variables ξ, η.

Thus the independent solution of the hydrodynamic equations with $\delta p \neq 0$ is characterized by condition of a potential and isentropic character, where δp and φ satisfy Eq. (1.4).

The position of the characteristics relative to the shock wave is shown in Fig. 1. As in the previous paper, the illustrations are shown for a gas with $\gamma = 1.4$, in which the pressure increases by a factor of 10 in the shock. However, the horizontal velocity v_x is less and amounts to 0.5 of the velocity of sound in front of the wave (for $y < 0$). Here, for $y < 0$, $c/v = 0.333$, and the flow is supersonic; for $y < 0$: $c/v = 1.75$ and the flow is subsonic. From the side $y < 0$, any of the perturbations calculated in Ref. 1 can fall on the shock wave. As a result, the entropy-vortical perturbation arises for $y > 0$, and is propagated along the characteristics; the perturbation is describe by the solution of the elliptic equations. To speak here of perturbations "arriving" from the side $y > 0$ has no meaning because of the elliptic character of the equations. For a given type of perturbation incident from the side $y < 0$, the solution of the problem will be a linear combination of independent solutions that are determined from the boundary conditions on the discontinuity.

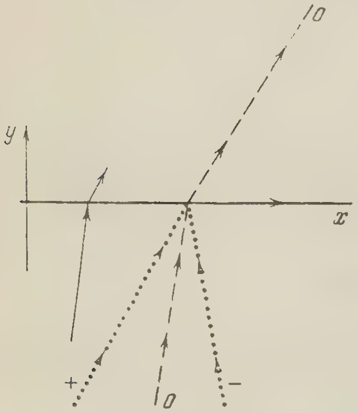

Fig. 1.———— shock wave, --- — characteristics of the entropy-vortical perturbation, which coincide with the flow lines, ... — characteristics of the acoustic perturbation,

The boundary conditions on the discontinuity were derived in Ref. 1. They are:

$$\delta v_{0x} - \delta v_x = (v_y - v_{0y})\,\eta'; \tag{1.9}$$

$$\frac{\delta v_{0y} - \delta v_y}{v_{0y} - v_y} = \frac{1}{2}\left[\frac{\delta p - \delta p_0}{p - p_0} + \frac{\delta V_0 - \delta V}{V_0 - V}\right]; \tag{1.10}$$

$$\delta V = \frac{\partial V}{\partial p}\,\delta p + \frac{\partial V}{\partial p_0}\,\delta p_0 + \frac{\partial V}{\partial V_0}\,\delta V_0; \tag{1.11}$$

$$\frac{2}{v_{0y}}(\delta v_{0y} - v_{0x}\eta') = \frac{2\delta V_0}{V_0} + \frac{\delta p - \delta p_0}{p - p_0} - \frac{\delta V_0 - \delta V}{V_0 - V}. \tag{1.12}$$

Furthermore, the solution of the elliptical equations must satisfy the condition:

$$\delta p = 0; \ \nabla\varphi = 0 \text{ at infinity} \tag{1.13}$$

2. SOLUTION OF THE EQUATIONS

Let us proceed to finding the solutions (of interest to us) set up by the incident perturbation.

To find δp and φ, it is necessary to solve Laplace's equation with boundary conditions on the shock wave ($y = \eta = 0$).

Transforming the boundary conditions (1.9) — (1.12) in the same way as in Ref. 1, we get

$$2\cot\varphi\,(v_y\delta v_x - v_x\delta v_y) - \left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\delta p = F(x); \tag{2.3}$$

$$\begin{aligned} F \equiv 2\cot\varphi(v_y\delta v_{0x} - v_x\delta v_{0y}) - \left[V_0\cot^2\varphi\left(1 + j^2\frac{\partial V}{\partial p_0}\right) - V\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0 + 2j\frac{V}{V_0}(V_0 - V)\,\delta v_{0y} \\ + j^2\left[\left(1 - \frac{\partial V}{\partial V_0}\right)(V + V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0 - V)\right]\delta V_0. \end{aligned} \tag{2.4}$$

Now, differentiating (2.3) with respect to x (along the shock wave) and, making use of the equations of hydrodynamics, we get

$$v_x\frac{\partial\delta v_y}{\partial x} - v_y\frac{\partial\delta v_x}{\partial x} = \frac{v_xv_y}{c^2}V\frac{\partial\delta p}{\partial x} - \left(1 - \frac{v_y^2}{c^2}\right)V\frac{\partial\delta p}{\partial y}.$$

As a result, we obtain the boundary condition in which there enters only the derivatives of δp along the shock wave and normal to it; transforming to ξ, η in it, we finally get

$$-\left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\frac{\partial\delta p}{\partial\xi} + 2V\cot\varphi\sqrt{1 - M^2}\frac{\partial\delta p}{\partial\eta} = \frac{\partial F}{\partial\xi}. \tag{2.5}$$

To find the harmonic functions which satisfy (2.5), we introduce the quantity Φ by the definition

$$\Phi = \partial\delta p/\partial\eta + i\,\partial\delta p/\partial\xi. \tag{2.6}$$

By virtue of Laplace's equation for δp the quantity Φ, which is considered as a function of the complex variable

$$\zeta = \xi + i\eta = \frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}}x + \left(\frac{v_xv_y/c^2}{\sqrt{1 - M^2}} + i\right)y, \tag{2.7}$$

is analytic in the upper half plane.

Thus the problem reduces to finding an analytic function along the given connection of its imaginary and real parts for $\eta = 0$ (the Riemann–Hilbert problem). If we now define the function $\Omega(\zeta)$ by the expression

$$\Omega(\zeta) = \int^{\zeta}\Phi(\zeta)\,d\zeta,$$

where the integration is carried out along any contour lying in the upper half plane and diverging to infinity, then δp is connected with $\Omega(\zeta)$ by the obvious relation

$$\delta p(\xi, \eta) = \operatorname{Im}\Omega(\zeta). \tag{2.8}$$

For the determination of δv, we note that $\delta v = \delta v^{(1)} + \delta v^{(2)}$, where $\delta v^{(1)}$ is determined by the entropy-vortical solution, and $\delta v^{(2)} = \nabla\varphi$ by the solution of the elliptic equations. It is easy to see that a determination of $\delta v^{(2)}$ reduces again to a solution of the Riemann–Hilbert problem. Substituting in Eq. (2.3) for δp its expression in terms of $\delta v^{(2)}$ [Eq. (1.7)] and in place of δv the sum $\delta v^{(1)} + \delta v^{(2)}$, we find that $\delta v^{(1)}$ drops out of the expression and we obtain the following relation between the components $\delta v_x^{(2)}$ and $\delta v_y^{(2)}$ on the shock wave:

$$j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi)\,\delta v_y^{(2)}-j\cot\varphi\left[(V-V_0)+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)\right]\delta v_x^{(2)}=-F(x). \tag{2.9}$$

Transforming to the variables ξ, η, and introducing the notation

$$\delta v_\xi=\partial\varphi/\partial\xi,\quad \delta v_\eta=\partial\varphi/\partial\eta;$$

$$\delta v_x^{(2)}=\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}\,\delta v_\xi,\quad \delta v_y^{(2)}=\delta v_\eta+\frac{v_xv_y/c^2}{\sqrt{1-M^2}}\,\delta v_\xi,$$

we finally obtain:

$$\begin{aligned}j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi)\,\delta v_\eta-\frac{j}{\sqrt{1-M^2}}\Big\{\cot\varphi\left[(V-V_0)+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)\right]\\ -\frac{j^3V}{c^2}[V\cot\varphi(V-V_0)+2V_0(V+V_0\cot^2\varphi)]\Big\}\,\delta v_\xi=-F(\xi).\end{aligned} \tag{2.10}$$

From this it is evident that the function

$$\Psi=\delta v_\eta+i\delta v_\xi,$$

considered as a function of the complex variable ζ, is analytic in the upper half plane; the determination of $\Psi(\zeta)$ also reduces to the Riemann–Hilbert problem.

The components of $\delta v^{(2)}$ are connected with $\Psi(\zeta)$ by the following obvious relations:

$$\delta v_x^{(2)}=(1-v_y^2/c^2)(1-M^2)^{-1/2}\,\mathrm{Im}\,\Psi(\zeta),\quad \delta v_y^{(2)}=\mathrm{Re}\,\Psi(\zeta)+(v_xv_y/c^2)(1-M^2)^{-1/2}\,\mathrm{Im}\,\Psi(\zeta). \tag{2.11}$$

As is well known,[2] the solution of the Riemann–Hilbert problem for the function $\Phi(\zeta)=u(\xi,\eta)+iv(\xi,\eta)$ satisfying the condition $au-bv=f(\xi)$ on the line $\eta=0$, where a and b are constants, and $f(\xi)$ is an arbitrary function, is given by the following integral of the Cauchy type:

$$\Phi(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{f(\tau)\,d\tau}{\tau-\zeta}.$$

Therefore, for the function $\Phi(\zeta)$ of interest to us, we have:

$$\Phi(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{\partial F(\tau)}{\partial\tau}\frac{d\tau}{\tau-\zeta},\qquad a=2V\cot\varphi\sqrt{1-M^2},\quad b=V_0\cot^2\varphi\left(1-j^2\frac{\partial V}{\partial p}\right)-V\left(1+j^2\frac{\partial V}{\partial p}\right). \tag{2.12}$$

We shall consider that the function $F\to 0$ for $\tau\to\pm\infty$. Then, at infinity, $\Phi(\zeta)$ behaves as $1/\zeta^2$, so that the integral $\int\Phi(\zeta)\,d\zeta$ along the contour which goes off to infinity does have meaning; in this case,

$$\Omega(\zeta)=\int^{\zeta}\Phi(\zeta)\,d\zeta=-\frac{b+ia}{\pi(a^2+b^2)}\int d\zeta\int\frac{dT}{\tau-\zeta}\frac{\partial F(\tau)}{\partial\tau}.$$

Reversing the order of integration in the latter integral, and integrating over τ by parts, we find

$$\Omega(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{d\tau}{(\tau-\zeta)^2}\frac{\partial F(\tau)}{\partial\tau}=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}. \tag{2.13}$$

The latter integration by parts is valid in the case in which $\partial F(\tau)/\partial\tau$ has finite discontinuities. Thus

$$\delta p(x,y)=-\mathrm{Im}\,\frac{b+ia}{\pi(a^2+b^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}=-\frac{1}{\pi(a^2+b^2)}\int\frac{F(t)}{G(t,x,y)}\left[a\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\,y\right)+b\,\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}\,y\right]dt, \tag{2.14}$$

where we use the notation

$$G(t,x,y)=\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\right)y^{2}+\frac{c^4(1-M^2)}{(c^2-v_y^2)^2}\,y^2.$$

In what follows we need the value of $\delta p(x,y)$ on the shock itself. As is well known (see, for example, Ref. 2), the value of the integral (2.13) at any point ξ on the boundary is equal to

$$\lim_{\zeta\to\xi}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}=\pi iF(\xi)+\int\frac{F(\tau)\,d\tau}{\tau-\xi}, \tag{2.15}$$

where the diverging integral on the right side is taken in the sense of its principal value. We then have

$$\delta p(x,\ 0)=-\frac{1}{\pi(a^2+b^2)}\left[b\pi F(x)+a\int\frac{F(t)\,dt}{t-x}\right]. \tag{2.16}$$

In similar fashion, we have for $\Psi(\zeta)$:

$$\Psi(\zeta)=\frac{\beta+i\alpha}{\pi(\alpha^2+\beta^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta},\quad \alpha=j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi),$$
$$\beta=j(1-M^2)^{-1/2}\left\{\cot\varphi\left[V-V_0+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)-\frac{j^3V}{c^2}[V\cot\varphi(V-V_0)+2V_0(V+V_0\cot^2\varphi)]\right\}. \tag{2.17}$$

The components of $\delta v^{(2)}$, in accord with (2.11), are expressed by the following formulas:

$$\delta v_x^{(2)}(x,y)=\frac{1-v_y^2/c^2}{\pi\sqrt{1-M^2}(\alpha^2+\beta^2)}\int\frac{F(t)}{G(t,\ x,\ y)}\left[\alpha\left(t-x-\frac{v_xv_y}{c^2-v_y^2}y\right)+\beta\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}y\right]dt,$$
$$\delta v_y^{(2)}(x,\ y)=\frac{v_xv_y}{c^2-v_y^2}\delta v_x^{(2)}+\frac{1}{\pi(\alpha^2+\beta^2)}\int\frac{F(t)}{G(t,\ x,\ y)}\left[\beta\left(t-x-\frac{v_xv_y}{c^2-v_y^2}y\right)-\alpha\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}y\right]dt. \tag{2.18}$$

We take the values of $\delta v_x^{(2)}$ and $\delta v_y^{(2)}$ on the shock wave itself:

$$\delta v_x^{(2)}(x,\ 0)=\frac{1-v_y^2/c^2}{\pi\sqrt{1-M^2}(\alpha^2+\beta^2)}\left[\pi\beta F(x)+\alpha\int\frac{F(t)\,dt}{t-x}\right],$$
$$\delta v_y^{(2)}(x,\ 0)=\frac{1}{\pi(\alpha^2+\beta^2)}\left[-\pi\alpha F(x)+\beta\int\frac{F(t)\,dt}{t-x}\right]+\frac{v_xv_y}{c^2-v_y^2}\delta v_x^{(2)}(x,\ 0). \tag{2.19}$$

We now proceed to finding the contribution of the entropy-vortical solution. The value of δV consists of two parts:

$$\delta V=\delta V_1+\delta V_2,\ \delta V_1=(\partial V/\partial s)_p\,\delta s,\ \delta V_2=-(V/c)^2\,\delta p.$$

Condition (1.11) gives for δV_1 on the shock wave:

$$\delta V_1=\left(\frac{V^2}{c^2}+\frac{\partial V}{\partial p}\right)\delta p+\frac{\partial V}{\partial p_0}\delta p_0+\frac{\partial V}{\partial V_0}\delta V_0. \tag{2.20}$$

If we then substitute the expression for δp [Eq. (2.16)] and replace the argument x by $x-y/k$, we obtain:

$$\delta V_1(x,\ y)=-\frac{V^2/c^2+\partial V/\partial p}{\pi(a^2+b^2)}\left[b\pi F\left(x-\frac{y}{k_0}\right)+a\int\frac{F(t)\,dt}{t-x+y/k_0}\right]+\frac{\partial V}{\partial p_0}\delta p_0\left(x-\frac{y}{k_0}\right)+\frac{\partial V}{\partial V_0}\delta V_0(x-y/k_0). \tag{2.21}$$

The divergent integral here, as in subsequent formulas, is understood in the sense of the principal value.

The contribution of the entropy-vortical solution to the velocity δv_1, is determined by the quantity $I_0 = v\delta v_1$. Repeating the work of the previous paper, we obtain the value of I_0 on the shock wave:

$$I_0=v_0\delta v_0-\frac{1}{2}(V_0-V)\left(1+j^2\frac{\partial V}{\partial p}\right)\delta p+\frac{1}{2}\left[V\left(1+j^2\frac{\partial V}{\partial p_0}\right)+V_0\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0+\frac{1}{2}j^2(V_0-V)\left[2\frac{V}{V_0}-1+\frac{\partial V}{\partial V_0}\right]\delta V_0. \tag{2.22}$$

Replacing δp by its value (2.16), replacing the argument x by $x-y/k_0$, we find

$$I_0(x,\ y)=v_0\delta v_0(x-y/k_0)+\frac{1}{2}\frac{(V_0-V)(1+j^2\partial V/\partial p)}{\pi(a^2+b^2)}\left[b\pi F\left(x-\frac{y}{k_0}\right)+a\int\frac{F(t)\,dt}{t-x+y/k_0}\right]$$
$$+\frac{1}{2}\left[V\left(1+j^2\frac{\partial V}{\partial p_0}\right)+V_0\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0\left(x-\frac{y}{k_0}\right)+\frac{1}{2}j^2(V_0-V)\left[2\frac{V}{V_0}-1+\frac{\partial V}{\partial V_0}\right]\delta V_0\left(x-\frac{y}{k_0}\right). \tag{2.23}$$

We recall that δv_1 is connected with I_0 by the relation

$$\delta v_1=(v/M^2c^2)\,I_0(x,\ y).$$

The complete solution is obtained by summation of the entropy-vortical solution and the solution of the elliptic equation.

Finally, let us compute the form of the perturbed sound wave which we shall characterize by the curvature $K(x) \approx \eta''(x)$, for which we make use of the condition (1.12). Differentiating with respect to x and taking it into account that

$$\frac{\partial \delta p}{\partial x} = \frac{1 - v_y^2 c^2}{\sqrt{1 - M^2}} \operatorname{Im} \Phi(\zeta),$$

we find

$$K(x) \Rightarrow \eta''(x) = \tan\varphi \left\{ \frac{\delta v'_{0y}(x)}{v_{0y}} + \frac{V/V_0 - \partial V/\partial V_0}{V_0 - V} \delta V'_0(x) + \frac{1 - j^2 \partial V/\partial p_0}{2(p - p_0)} \delta p'_0(x) - \frac{1 + j^2 \partial V/\partial p}{2(p - p_0)} \frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}} \operatorname{Im} \Phi \left(\frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}} x \right) \right\}. \tag{2.24}$$

We see that the curvature of the front of the shock wave is determined by the first derivatives of hydrodynamic quantities.

The formulas obtained in this section have a general character, inasmuch as no assumptions have been made regarding the character of the incident perturbation. If we set

$$\delta s_0 = 0;\ \delta V_0 = -\frac{V_0^2}{c_0^2} \delta p_0,\ \delta v_{0x} = \frac{\mp v_{0y} \sqrt{M_0^2 - 1} - v_{0x}}{M_0^2 c_0^2} V_0 \delta p_0, \quad \delta v_{0y} = \frac{\pm v_{0x} \sqrt{M_0^2 - 1} - v_{0y}}{M_0^2 c_0^2} V_0 \delta p_0, \tag{2.25}$$

this will mean that the incident perturbation is of an acoustic type and is connected with the characteristics k_{+0} (upper sign; the shock wave "overtakes" the perturbation) or k_{-0} (lower sign: the shock wave "encounters" the perturbation).

If we are interested in the interaction of the shock wave with the entropy-vortical perturbation, we must set

$$\delta p_0 = 0;\ \delta V_0 = (\partial V/\partial s)_{p_0} \delta s_0;\ v_{0x} \delta v_{0y} - v_{0y} \delta v_{0x} = 0. \tag{2.26}$$

This perturbation is connected with the characteristics k_{00}.

3. INTERACTION OF A WEAK DISCONTINUITY WITH A SHOCK WAVE

We now proceed to a study of the interaction of a weak discontinuity with a shock wave, for which we will be interested in the intensity of a weak tangential discontinuity which passes "through" the shock wave, the behavior of all quantities near the point of intersection of the weak discontinuity and the shock wave, and also the form of the shock wave close to the point of intersection.

Let a small perturbation of arbitrary type, with a weak discontinuity, fall on the shock wave from the side $y < 0$; we consider the most interesting case, in which the weak discontinuity is a discontinuity of derivatives of hydrodynamical quantities. The intensity of the incident weak discontinuity is characterized by the jumps Δ, Δ_1, and Δ_2 in the derivatives of the pressure (for weak discontinuities of the acoustic type), of the entropy Δ_1, and of the quantity I_0 (for weak tangential discontinuities) respectively along the normal to the characteristic bearing the discontinuity. We choose the point of intersection of the discontinuity with the shock wave as the origin of the coordinates. We denote the jumps in the derivatives of the hydrodynamical quantities at the origin, along the shock wave, as Δp_0, Δv_0, etc. In correspondence with what has been shown, the decomposition of all the quantities on the shock wave from the side $y < 0$ near the origin will have the form

$$f(x) = \begin{cases} f(0) + (A_f - \Delta_f)x & \text{for } x < 0, \\ f(0) + A_f x & \text{for } x > 0. \end{cases}$$

Let us determine tentatively the behavior of integrals of the Cauchy type, which are encountered in our problem, near the point $\zeta = 0$.

As is well known,[1] the integral (2.12), in the neighborhood of the point $\zeta = 0$, and for the presence of a discontinuity in $\partial F/\partial \tau$, will be logarithmic:

$$\Phi(\zeta) \simeq \frac{b + ia}{\pi(a^2 + b^2)} \frac{\sqrt{1 - M^2}}{1 - v_y^2/c^2} \Delta_F (\ln \zeta + A), \tag{3.1}$$

where A is a constant dependent on the behavior of the function $F(\tau)$ everywhere on the x axis.

In what follows, we need the values of (3.1) on the real axis

$$\Phi(\xi) \simeq \begin{cases} \dfrac{\Delta_F}{\pi(a^2+b^2)}[(b\ln|\xi|-a\pi)+i(a\ln|\xi|+b\pi)+A] & \text{for } \xi<0 \\ \dfrac{\Delta_F}{\pi(a^2+b^2)}[(b+ia)\ln\xi+A] & \text{for } \xi>0. \end{cases} \tag{3.2}$$

For the determination of the behavior of the function $\Omega(\zeta)$ close to the origin, we integrate (3.1):

$$\Omega(\zeta)=\Omega(0)+\frac{(b+ia)\Delta_F\sqrt{1-M^2}}{\pi(a^2+b^2)(1-v_y^2/c^2)}\zeta(\ln\zeta+A-1). \tag{3.3}$$

This formula gives an expression for δp behind the shock wave in the region about the origin. Actually, taking the imaginary part of the expression (3.3), and transforming to the original variables x, y, we will have

$$\begin{aligned}\delta p(x,y)=\delta p(0,0)+\frac{\sqrt{1-M^2}}{1-v_y^2/c^2}\,\frac{\Delta_F}{\pi(a^2+b^2)}\Bigg[&\left(a\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x+\left(\frac{av_xv_y/c^2}{\sqrt{1-M^2}}+b\right)y\right)\frac{1}{2}\ln\left[\left(x+\frac{v_xv_y}{c^2-v_y^2}y\right)^2+\frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right]\\ &+\left(b\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x+\left(\frac{bv_xv_y/c^2}{\sqrt{1-M^2}}-a\right)y\right)\tan^{-1}\frac{\sqrt{1-M^2}\,y}{(1-v_y^2/c^2)x+(v_xv_y/c^2)y}\Bigg]+A_px+B_py.\end{aligned} \tag{3.4}$$

Here $\delta p(0,0)$ is determined by Eq. (2.16); A_p and B_p are constants which are connected in a definite way with A. On the shock wave itself, the following relations hold:

$$\begin{aligned}\delta p(x,0)&=\delta p(0,0)+\frac{\Delta_Fa}{\pi(a^2+b^2)}x\ln|x|+(A_p-\Delta_p)x && \text{for } x<0,\\ \delta p(x,0)&=\delta p(0,0)+\frac{\Delta_Fa}{\pi(a^2+b^2)}x\ln x+A_px && \text{for } x>0,\end{aligned} \tag{3.5}$$

where

$$\Delta_p=-\Delta_Fb/(a^2+b^2).$$

The function $\Psi(\zeta)$ which is necessary for the calculation of $\delta v^{(2)}$ differs from $\Omega(\zeta)$ by the replacement of a by $-\alpha$ and of b by $-\beta$. Making use of Eqs. (2.11), we find (close to the origin):

$$\begin{aligned}\delta v_x^{(2)}(x,y)=\delta v_x^{(2)}(0,0)-\frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\Bigg[&\left(\alpha\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x+\left(\frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}+\beta\right)y\right)\frac{1}{2}\ln\left[\left(x+\frac{v_xv_y}{c^2-v_y^2}y\right)^2+\frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right]\\ &+\left(\beta\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x+\left(\frac{\beta v_xv_y/c^2}{\sqrt{1-M^2}}-\alpha\right)y\right)\tan^{-1}\frac{y\sqrt{1-M^2}}{(1-v_y^2/c^2)x+(v_xv_y/c^2)y}\Bigg]+A_{v_x}x+B_{v_x}y.\\ \delta v_y^{(2)}(x,y)=\delta v_y^{(2)}(0,0)+\frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\cdot\frac{\sqrt{1-M^2}}{1-v_y^2/c^2}\Bigg\{&\Bigg[\left(\alpha\left(1-\frac{v_x^2v_y^2/c^4}{1-M^2}\right)-\frac{2\beta v_xv_y/c^2}{\sqrt{1-M^2}}\right)y-\left(\beta+\frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x\Bigg]\frac{1}{2}\\ &\times\ln\left[\left(x+\frac{v_xv_y}{c^2-v_y^2}y\right)^2+\frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right]+\Bigg[\left(\alpha-\frac{\beta v_xv_y/c^2}{\sqrt{1-M^2}}\right)\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x+\Bigg(\frac{2\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\\ &+\beta\left(1-\frac{v_x^2v_y^2/c^4}{1-M^2}\right)\Bigg)y\Bigg]\tan^{-1}\frac{y\sqrt{1-M^2}}{(1-v_y^2/c^2)x+(v_xv_y/c^2)y}\Bigg\}+A_{v_y}x+B_{v_y}y.\end{aligned} \tag{3.6}$$

On the shock wave itself, we have

$$\begin{aligned}\delta v_x^{(2)}(x,0)&=\delta v_x^{(2)}(0,0)-\frac{\Delta_F\alpha}{\pi(\alpha^2+\beta^2)}\left(1-\frac{v_y^2}{c^2}\right)x\ln|x|+(A_{v_x}-\Delta_{v_x})x && \text{for } x<0,\\ \delta v_x^{(2)}(x,0)&=\delta v_x^{(2)}(0,0)-\frac{\Delta_F\alpha}{\pi(\alpha^2+\beta^2)}(1-v_y^2/c^2)x\ln x+A_{v_x}x && \text{for } x>0,\\ \delta v_y^{(2)}(x,0)&=\delta v_y^{(2)}(0,0)-\frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\left(\beta+\frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)x\ln|x|+(A_{v_y}-\Delta_{v_y})x, && \text{for } x<0,\\ \delta v_y^{(2)}(x,0)&=\delta v_y^{(2)}(0,0)-\frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\left(\beta+\frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)x\ln x+A_{v_y}x && \text{for } x>0,\end{aligned}$$

where

$$\Delta_{v_x} = \frac{\Delta_F \beta}{\alpha^2 + \beta^2} \frac{1 - v_y^2 / c^2}{\sqrt{1 - M^2}}, \quad \Delta_{v_y} = \frac{\Delta_F}{\alpha^2 + \beta^2} \left(\frac{\beta v_x v_y / c^2}{\sqrt{1 - M^2}} - \alpha \right).$$

We now find the contribution of the entropy-vortical solution. The quantity $\delta V^{(1)}$ is determined from Eq. (2.20); substituting $x - y/k_0$ for x in this expression, we get the value of $\delta V^{(1)}$ in the vicinity of the flow line $x - y/k_0 = 0$, which will be the bearer of a real tangential discontinuity of very unique character:

$$\delta V^{(1)}(x, y) = \delta V^{(1)}(0, 0) + \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p} \right) \frac{\Delta_F a}{\pi (a^2 + b^2)} \left(x - \frac{y}{k_0} \right) \ln \left(\frac{y}{k_0} - x \right) + (A_V - \Delta_V) \left(x - \frac{y}{k_0} \right) \quad \text{for } x < y/k_0,$$

$$\delta V^{(1)}(x, y) = \delta V^{(1)}(0, 0) + \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p} \right) \frac{\Delta_F a}{\pi (a^2 + b^2)} \left(x - \frac{y}{k_0} \right) \ln \left(x - \frac{y}{k_0} \right) + A_V \left(x - \frac{y}{k_0} \right) \quad \text{for } x > y/k_0,$$

$$\Delta_V = \frac{\partial V}{\partial p_0} \Delta_{p_0} + \frac{\partial V}{\partial V_0} \Delta_{V_0} - \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p} \right) \frac{\Delta_F b}{a^2 + b^2}. \tag{3.8}$$

Of similar character is the special feature possessed by the quantity I_0. According to (2.2), we have

$$I_0(x, y) = I_0(0, 0) - \frac{1}{2} (V_0 - V) \left(1 + j^2 \frac{\partial V}{\partial p} \right) \frac{\Delta_F a}{\pi (a^2 + b^2)} \left(x - \frac{y}{k_0} \right) \ln \left(\frac{y}{k_0} - x \right) + (A_{I_0} - \Delta_{I_0}) \left(x - \frac{y}{k_0} \right) \quad \text{for } x < y/k_0,$$

$$I_0(x, y) = I_0(0, 0) - \frac{1}{2} (V_0 - V) \left(1 + j^2 \frac{\partial V}{\partial p} \right) \frac{\Delta_F a}{\pi (a^2 + b^2)} \left(x - \frac{y}{k_0} \right) \ln \left(x - \frac{y}{k_0} \right) + A_{I_0} \left(x - \frac{y}{k_0} \right) \quad \text{for } x > y/k_0,$$

$$\Delta_{I_0} = v_0 \Delta_{v_0} + \frac{1}{2} \left[V \left(1 + j^2 \frac{\partial V}{\partial p_0} \right) + V_0 \left(1 - j^2 \frac{\partial V}{\partial p_0} \right) \right] \Delta_{p_0}$$

$$+ \frac{1}{2} j^2 (V_0 - V) \left[2 \frac{V}{V_0} - 1 + \frac{\partial V}{\partial V_0} \right] \Delta_{V_0} + \frac{1}{2} (V_0 - V) \left(1 + j^2 \frac{\partial V}{\partial p} \right) \frac{\Delta_F b}{a^2 + b^2}. \tag{3.9}$$

We see from Eqs. (3.8) and (3.9) that a weak tangential discontinuity, which passes through a shock wave, differs in the subsonic case in an unusual character: in addition to the jump in the coefficients in the linear terms of the expansion of hydrodynamical quantities in the vicinity of a weak discontinuity, this expansion is characterized by the present of a logarithmic singularity.

For the calculation of the curvature of the shock wave in the vicinity of the origin, we make use of Eq. (2.24):

$$K(x) = \tan \varphi \left\{ A_K - \Delta_K - \frac{1}{2} \frac{1 + j^2 \frac{\partial V}{\partial p}}{p - p_0} \frac{\Delta_F a}{\pi (a^2 + b^2)} \ln |x| \right\} \quad \text{for } x < 0,$$

$$K(x) = \tan \varphi \left\{ A_K - \frac{1}{2} \frac{1 + j^2 \frac{\partial V}{\partial p}}{p - p_0} \frac{\Delta_F a}{\pi (a^2 + b^2)} \ln x \right\} \quad \text{for } x > 0,$$

$$\Delta_K = \frac{\Delta v_{0y}}{v_{0y}} + \frac{V / V_0 - \partial V / \partial V_0}{V_0 - V} \Delta_{V_0} + \frac{1 - j^2 \partial V / \partial p_0}{2 (p - p_0)} \Delta_{p_0} + \frac{1 + j^2 \partial V / \partial p}{2 (p - p_0)} \frac{\Delta_F b}{a^2 + b^2}.$$

Thus the curvature of the shock wave in the subsonic case also has a logarithmic singularity at the point of intersection with a weak discontinuity.

There now remains only the connecting of the quantities Δ_F, Δ_{p_0}, etc. with the intensity of the incident weak discontinuity. Let us first consider the case of the incidence of the usual weak discontinuity of the acoustic type. Inasmuch as the weak discontinuity can fall along either of two characteristics k_{+0} and k_{-0}, we must distinguish two cases (Figs. 2 and 3). The connection of the quantity Δ_{p_0} with the intensity of the incident weak discontinuity follows from elementary geometric considerations:

$$\Delta_{p_0} = \frac{k_{\pm 0}}{\sqrt{1 + k_{\pm 0}^2}} \Delta_{\pm} = \frac{v_{0x} v_{0y} \pm c_0^2 \sqrt{M_0^2 - 1}}{c_0 [v_{0x} \sqrt{M_0^2 - 1} \pm v_{0y}]} \Delta_{\pm}. \tag{3.11}$$

Therefore, with the help of Eqs. (2.25), we get

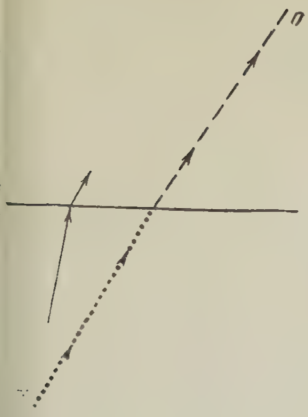

FIG. 2

$$\Delta_{V_0} = -V_0^2 \frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0^3[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm},$$

$$\Delta_{v_{0x}} = \frac{V_0}{M_0^2}\frac{[\mp v_{0y}\sqrt{M_0^2-1} - v_{0x}][v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}]}{c_0^3[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm}, \tag{3.12}$$

$$\Delta_{v_{0y}} = \frac{V_0}{M_0^2}\frac{[\pm v_{0x}\sqrt{M_0^2-1} - v_{0y}][v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}]}{c_0^3[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm}.$$

Further, we find:

$$\Delta_F = \left\{\frac{2j^2V_0}{M_0^2c_0^2}[(V_0-V)(V_0\cot^2\varphi - V) \mp (V^2+V_0^2\cot^2\varphi)\cot\varphi\sqrt{M_0^2-1}]\right.$$
$$-\left[V_0\cot^2\varphi\left(1+j^2\frac{\partial V}{\partial p_0}\right) - V\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right] - \frac{j^2V_0^2}{c_0^2}\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi)\right.$$
$$\left.\left.-2\frac{V}{V_0}(V_0-V)\right]\right\}\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm};$$

$$\Delta_V = -\left(\frac{V^2}{c^2}+\frac{\partial V}{\partial p}\right)\frac{b}{a^2+b^2}\Delta_F + \left(\frac{\partial V}{\partial p_0} - \frac{V_0^2}{c_0^2}\frac{\partial V}{\partial V_0}\right)\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm};$$

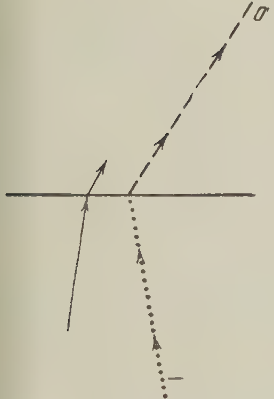

FIG. 3

$$\Delta_{I_0} = \frac{1}{2}(V_0-V)\left(1+j^2\frac{\partial V}{\partial p}\right)\frac{b}{a^2+b^2}\Delta_F - \frac{1}{2}(V_0-V)\left[1+j^2\frac{\partial V}{\partial p_0}\right.$$
$$\left.+\frac{j^2V_0^2}{c_0^2}\left(2\frac{V}{V_0} - 1 + \frac{\partial V}{\partial V_0}\right)\right]\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm};$$

$$\Delta_K = \frac{1+j^2\dfrac{\partial V}{\partial p}}{2(p-p_0)}\frac{b}{a^2+b^2}\Delta_F + \left[\frac{1-j^2\partial V/\partial p_0}{2(p-p_0)} - \frac{V/V_0 - \partial V/\partial V_0}{V_0-V}\frac{V_0^2}{c_0^2}\right. \tag{3.13}$$
$$\left.+\frac{\pm\cot\varphi\sqrt{M_0^2-1}-1}{M_0^2c_0^2}\right]\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\Delta_{\pm}.$$

The upper and lower signs in these equations refer to the cases a and b, respectively.

Now let us consider the case of the interaction of a weak tangential discontinuity with the shock wave (Fig. 4). In correspondence with Eqs. (2.26), we have

$$\Delta_{p_0} = 0; \quad \Delta_{V_0} = \frac{v_{0y}}{M_0c_0}\left(\frac{\partial V}{\partial s}\right)_{p_0}\Delta_1; \qquad \Delta_{v_{0x}} = \frac{v_{0x}v_{0y}}{M_0^3c_0^3}\Delta_2; \quad \Delta_{v_{0y}} = \frac{v_{0y}^2}{M_0^3c_0^3}\Delta_2. \tag{3.14}$$

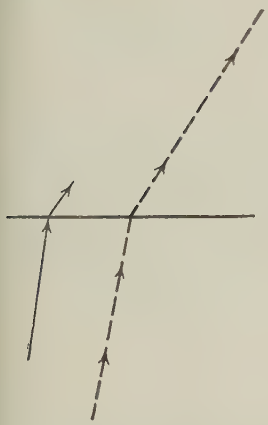

FIG. 4

Therefore, with the help of elementary calculations, we obtain

$$\Delta_F = 2j^3(V_0-V)V_0(V-V_0\cot^2\varphi)\frac{\Delta_2}{M_0^3c_0^3} + \frac{j^3}{M_0c_0}\left[\left(1-\frac{\partial V}{\partial V_0}\right)(V+V_0\cot^2\varphi)\right.$$
$$\left.-2\frac{V}{V_0}(V_0-V)\right]V_0\left(\frac{\partial V}{\partial s}\right)_{p_0}\Delta_1; \quad \Delta_V = -\left(\frac{V^2}{c^2}+\frac{\partial V}{\partial p}\right)\frac{b}{a^2+b^2}\Delta_F + \frac{jV_0}{M_0c_0}\left(\frac{\partial V}{\partial s}\right)_{p_0}\frac{\partial V}{\partial V_0}\Delta_1;$$
$$\Delta_{I_0} = \frac{1}{2}(V_0-V)\left(1+j^2\frac{\partial V}{\partial p}\right)\frac{b}{a^2+b^2}\Delta_F + \frac{jV_0}{M_0c_0}\Delta_2 + \frac{1}{2}j^3(V_0-V)\left(2\frac{V}{V_0} - 1\right.$$
$$\left.+\frac{\partial V}{\partial V_0}\right)\frac{V_0}{M_0c_0}\left(\frac{\partial V}{\partial s}\right)_{p_0}\Delta_1; \quad \Delta_K = \frac{1+j^2\partial V/\partial p}{2(p-p_0)}\frac{b}{a^2+b^2}\Delta_F + \frac{jV_0}{M_0^3c_0^3}\Delta_2 + \frac{V/V_0 - \partial V/\partial V_0}{V_0-V}\frac{jV_0}{M_0c_0}\left(\frac{\partial V}{\partial s}\right)_{p_0}\Delta_1.$$

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TIME REVERSAL AND REACTIONS INVOLVING POLARIZED PARTICLES

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If the interaction between particles in a reversible process such as $a + b \rightleftarrows c + d$ is invariant under time reversal, one can obtain several relations between the angular distributions, polarization vectors, and polarization tensors of the particles in the direct and inverse reactions (for arbitrary particle spins). These relations may be considered a generalization of the so-called "principle of detailed balance."

WE shall consider reactions such as $a + b \rightarrow c + d$ ("direct" reaction D) and its inverse (I) reaction $c + d \rightarrow a + b$, where a, b, c, and d are "elementary" particles or nuclei with arbitrary spins. We assume that the physical system (consisting of either pair a and b or c and d) is invariant under time reversal.[1,2]

Experiments on such reactions give information on their S matrix. In order to find all the elements of the S matrix we must perform a complete set of experiments, which involves measuring not only the angular distributions of the reaction products, but also their polarization vectors and tensors in experiments with a polarized beam and a polarized target. In elastic reactions of the type $a + b \rightarrow a + b$, invariance under time reversal makes it possible to decrease the necessary complete set, just as do the conservation laws for angular momentum, etc. It will be shown in the present work (see Sec. 3), for instance, that once having measured the angular distribution for an unpolarized beam and an unpolarized target, we need measure only the polarization vectors of the products of the reaction in the next experiment. In this experiment the initial beam and target are polarized so that only their polarization vectors are nonzero. It is not necessary to measure also the polarization tensors (of rank higher than one), since what such a measurement would tell us about the S matrix will be obtained from other experiments of the complete set. In these other experiments the beam and target are polarized in a more complicated way (with nonzero polarization tensors). In general, we can conclude the following: it is possible to eliminate experiments in which one measures polarization tensors of rank higher than the highest rank polarization tensor describing the spin state of the beam and target.

For a reaction of the type $a + b \rightarrow c + d$, time reversal invariance leads only to the conclusion that an investigation of the direct reaction can be replaced by an investigation of the inverse reaction.

1. The results of this work will be given in terms of the statistical tensors defined previously.[3] We present here the relation between the components $\rho(q, \nu)$, with $\nu = -q, -q+1, \ldots, +q$, of a statistical tensor of rank q (with $q = 0, 1, \ldots, 2i$) of a single particle with spin i and the density matrix $(m_1|\rho|m_2)$ describing the spin state of an ensemble of such particles:

$$\rho(q, \nu) = \sqrt{2i+1} \sum_{m_1, m_2} (-1)^{i-m_2} (i i m_1 - m_2 | q\nu)(m_1|\rho|m_2).$$

The symbol $(i_1 i_2 m_1 m_2 | q\nu) = C^{q\nu}_{i_1 m_1 i_2 m_2}$ is a Clebsch-Gordan coefficient, and the rest of the notation is consistent with the previously cited work by the author.[3]

In the $a + b \rightarrow c + d$ reaction we obtain the following expression for the statistical tensor of the particles c and d in terms of the "scattering matrix" R (in the representation in which the spin projection and momentum are diagonal) and the statistical tensor of the beam a and target b (the notation is that used previously[3]):

$$\rho'(\mathbf{p}_c, \alpha'; q_c\tau_c q_d\tau_d) = [(2i_c+1)(2i_d+1)]^{1/2} [(2i_a+1)(2i_b+1)]^{-1/2}$$
$$\times \sum_{q_a\tau_a q_b\tau_b} (\mathbf{p}_c, \alpha', \alpha'; q_c\tau_c q_d\tau_d | W_D | \mathbf{p}_a, \alpha_1, \alpha_2; q_a\tau_a q_b\tau_b) \rho_{\alpha_{11}, \alpha_2}(\mathbf{p}_a; q_a\tau_a q_b\tau_b); \quad (1)$$

$$(\mathbf{p}_c, \alpha_1', \alpha_2'; q_c\tau_c q_d\tau_d \,|\, W_{\mathrm{D}} \,|\, \mathbf{p}_a, \alpha_1, \alpha_2; q_a\tau_a q_b\tau_b) = \sum_{m_c, m_d, m_c', m_d'} (-1)^{i_c - m_c'} (i_c i_c m_c - m_c' \,|\, q_c\tau_c)(-1)^{i_d - m_d'} (i_d i_d m_d - m_d' \,|\, q_d\tau_d) \times \tag{2}$$

$$\sum_{m_a, m_b, m_a', m_b'} (m_c m_d \mathbf{p}_c \alpha_1' \,|\, R \,|\, m_a m_b \mathbf{p}_a \alpha_1)(m_c' m_d' \mathbf{p}_c \alpha_2' \,|\, R \,|\, m_a' m_b' \mathbf{p}_a \alpha_2)^* (-1)^{-i+m_a'} (i_a i_a m_a - m_a' \,|\, q_a\tau_a)(-1)^{-i+m_b'} (i_b i_b m_b - m_b' \,|\, q_b\tau_b).$$

In what follows it is irrelevant whether all the spin indices refer to the same z axis of quantization, or whether τ_c, τ_d, m_c, and m_d are defined with respect to an axis of quantization[3] along $\mathbf{p}_c$, and τ_a, τ_b, m_a, and m_b are defined with respect to the $\mathbf{p}_a$ direction.

For the inverse reaction we may write two similar equations which we shall call (1′) and (2′) (the momentum of particles c and d in the rest system is $\mathbf{p}_c$; we are interested in particles a and b emitted in the $\mathbf{p}_a$ direction; the coordinate system is the same). Let us establish the relation between the elements $(m_a m_b \mathbf{p}_a \,|\, R \,|\, m_c m_d \mathbf{p}_c)$ in (2′) (the α indices are suppressed) and the corresponding elements of (2).

Invariance of this process under time reversal can be written[2]

$$(\Psi, \hat{S}\Phi) = (\Phi_K, \hat{S}\Psi_K) \tag{3}$$

(a similar relation holds for $\hat{R} = \hat{S} = 1$). According to the definition of the wave function of the time-reversed state (see Sec. 3 of Ref. 2) the reversed wave function of $\Psi_{m_a m_b \mathbf{p}}$ describes a state* in which the spin projections and momentum of a and b have the eigenvalues $-m_a$, $-m_b$, and $-\mathbf{p}$,† i.e.,

$$\Psi_{K m_a m_b \mathbf{p}} \approx \Psi_{-m_a, -m_b, -\mathbf{p}}$$

with an accuracy up to a phase factor whose expression is given by Huby[4] and Biedenharn,[5] namely

$$\Psi_{K m_a m_b \mathbf{p}} \equiv \hat{K}\Psi_{m_a m_b \mathbf{p}} = e^{i\beta}(-1)^{m_a + m_b}\Psi_{-m_a, -m_b, -\mathbf{p}}. \tag{4}$$

The phase β, which may depend on i_a and i_b and even on p will be unimportant in what follows. From the above we find that

$$(m_a m_b \mathbf{p}_a \,|\, R \,|\, m_c m_d \mathbf{p}_c)(m_a' m_b' \mathbf{p}_a \,|\, R \,|\, m_c' m_d' \mathbf{p}_c)^*$$

in (2′) can be replaced by

$$(-1)^{m_a + m_b + m_a' + m_b' + m_c + m_d + m_c' + m_d'} (-m_c, -m_d, -\mathbf{p}_c \,|\, R \,|\, -m_a, -m_b, -\mathbf{p}_a)(-m_c', -m_d', -\mathbf{p}_c \,|\, R \,|\, -m_a', -m_b', -\mathbf{p}_a)^*,$$

where the phase factor is $(-1)^{\tau_a + \tau_b + \tau_c + \tau_d}$, since $(-1)^{m_a' + m_b' + m_c' + m_d'} = (-1)^{-m_a' - m_b' - m_c' - m_d'}$ $(m_a' + m_b' + m_c' + m_d')$ is an integer and $\tau_a = m_a - m_a'$, etc. [see Eq. (2)].

After this we replace the indices m in (2′) by $-m$. This does not change the sums, since the indices m take on the values $-i, -i+1, \ldots, i-1, i$. The factors $(-1)^{\pm i \mp m}(iim - m \,|\, q\tau)$ in (2′) then become $(-1)^{\pm i \pm m}(ii - mm \,|\, q\tau)$, and according to the properties of the Clebsch-Gordan coefficients this is $(-1)^{-q}(-1)^{\mp i \pm m}(iim - m \,|\, q - \tau)$. We note that $(-1)^{i-m} = (-1)^{-i+m}$, since $i - m$ is an integer.

We finally find that all the coefficients

$$(\mathbf{p}_a; q_a\tau_a q_b\tau_b \,|\, W_{\mathrm{I}} \,|\, \mathbf{p}_c; q_c\tau_c q_d\tau_d)$$

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*The wave function $\Psi_{m_a m_b \mathbf{p}}$ describes the state of the physical system consisting of particles a and b (but not c and d) whose momentum in the rest system is p and whose spin projections are m_a and m_b.

†We assume that among the variables α (which represent all the variables other than p, m_a, and m_b characterizing the state of the system, e.g., intrinsic particle parity) there are none which change sign under time reversal. In addition, it is assumed for simplicity that the beam and target are not in mixed states with respect to α, or that the density matrix ρ in Eq. (1) contains the factor $\delta_{\alpha\alpha_1}\delta_{\alpha\alpha_2}$, and that after the reaction one measures the probabilities only for possible values of α (if they are measured at all) which can be given by matrix elements of ρ' diagonal in α.

of the inverse reaction are known if we know all the coefficients W_D of the direct $a + b \rightarrow c + d$ reaction, and vice versa:

$$(\mathbf{p}_a, \alpha;\, q_a\tau_a q_b\tau_b \,|\, W_0 \,|\, \mathbf{p}_c, \alpha';\, q_c\tau_c q_d\tau_d) = (-1)^{q_a+q_b+q_c+q_d+\tau_a+\tau_b+\tau_c+\tau_d}$$
$$\times (-\mathbf{p}_c, \alpha';\, q_c, -\tau_c, q_d, -\tau_d \,|\, W_D \,|\, -\mathbf{p}_a, \alpha;\, q_a, -\tau_a, q_b, -\tau_b). \tag{5}$$

2. The fundamental relation (5) can be obtained in a different way by expressing the invariance under time reversal in terms of the symmetry of the elements of the S (or R) matrix in the representation in which s, ℓ, J, E, and α are diagonal (see the previous footnote):[2,4]

$$(i_a i_b s l \alpha \,|\, R^{JE} \,|\, i_c i_d s' l' \alpha') = (i_c i_d s' l' \alpha' \,|\, R^{JE} \,|\, i_a i_b s l \alpha). \tag{3'}$$

Expressions for the statistical tensors of particles c and d in terms of such matrix elements and the statistical tensors of the beam and target have been given previously [Eqs. (2.10) and (2.11) of Ref. 3]; to be specific we shall use henceforth the second of these equations, (2.11). This can be written in the form of (1), where

$$(\mathbf{p}_c, \alpha';\, q_c\tau_c q_d\tau_d \,|\, W_D \,|\, \mathbf{p}_a, \alpha;\, q_a\tau_a q_b\tau_b) = N(4\pi)^{-2} \sum (-1)^{q'+\tau'} (q_c q_d \tau_c \tau_d \,|\, q'\tau')\, (2q_c+1)(2q_d+1)]^{1/2} X(i_c q_c i_c;\, s_1' q' s_2';\, i_d q_d i_d)$$
$$\times G^*_{\tau'}(J_1 l_1' s_1';\, Jq';\, J_2 l_2' s_2')\, (s_1' l_1' \alpha' \,|\, R^{J_1 E(p_a)} \,|\, s_1 l_1 \alpha)\, (s_2' l_2' \alpha' \,|\, R^{J_2 E(p_a)} \,|\, s_2 l_2 \alpha)^*$$
$$\times G_\tau(J_1 l_1 s_1;\, Jq;\, J_2 l_2 s_2)\, X(i_a q_a i_a;\, s_1 q s_2;\, i_b q_b i_b)\, [(2q_a+1)(2q_b+1)]^{1/2} (q_a q_b \tau_a \tau_b \,|\, q\tau)\, (-1)^{q+\tau} D^J_{\tau'\tau}(g_c g_a^{-1}). \tag{6}$$

Here $\sum$ denotes summation over q', s_1', s_2', ℓ_1', ℓ_2', J_1, J_2, ℓ_1, ℓ_2, s_1, s_2, q, and J. We have written $\tau' = \tau_c + \tau_d$ and $\tau = \tau_a + \tau_b$. The remaining notation is the same as that of the work cited.

We could now write out the expression [let us call it (6′)] for the coefficients W_I of the inverse $c + d \rightarrow a + b$ reaction (maintaining the same notation, i.e., using primed indices such as q', τ', ℓ', s' for particles c and d, and unprimed indices for a and b); then (3′) can be used to transform (6′), to the extent that this is possible, to the form of (6).

The Clebsch-Gordan coefficients, the X coefficients, and many other factors in the general term of the sum in (6′) can simply be permuted so that they occur in the same places as they do in (6). Further,

$$D^J_{\tau\tau'}(g_a g_c^{-1}) = (D^{-1})^J_{\tau\tau'}(g_c g_a^{-1}) = D^{J*}_{\tau'\tau}(g_c g_a^{-1}) = (-1)^{\tau-\tau'} D^J_{-\tau', -\tau}(g_c g_a^{-1}).$$

From Simon's[6] Eq. (B.1) for G_τ it follows that

$$G_\tau \backsim i^{l_1+l_2} \sum_L (l_1 l_2 00 \,|\, L0)\, (qJ\tau - \tau \,|\, L0)\, X = i^{l_1+l_2} \sum_L (l_1 l_2 00 \,|\, L0)\, (qJ - \tau\tau \,|\, L0)\, (-1)^{L-q-J} X \backsim (-1)^{-q-J} G^*_{-\tau}$$

Since $L + \ell_1 + \ell_2$ must be even [the presence of the coefficient $(\ell_1\ell_2 00 \,|\, \ell_1\ell_2 L0)$], we find that $(-1)^L = (-1)^{\ell_1+\ell_2}$, and $(-1)^{\ell_1+\ell_2} i^{\ell_1+\ell_2} = i^{-\ell_1-\ell_2}$. Thus $G^*_\tau G_{\tau'}$ in (6′) can be replaced by $(-1)^{q+q'} G^*_{-\tau'} G_{-\tau}$. Noting also that

$$(q_a q_b \tau_a \tau_b \,|\, q\tau) = (-1)^{q_a+q_b-q} (q_a q_b - \tau_a - \tau_b \,|\, q - \tau),$$

we finally obtain a relation which can be written in the form of (5) [or (8); see below].

It should be emphasized that in obtaining (5) by this method we started not only with a formulation of time-reversal invariance different than that of Sec. 1, but also assumed invariance under space rotations. This latter makes is possible to express the fundamental result in a form much more convenient for experiment. This involves rewriting (5) for the case in which the incident beams in the direct and inverse reactions have the same directions (usually chosen as the direction of the z axis) rather than two different directions $\mathbf{p}_a$ and $\mathbf{p}_c$. This is particularly convenient for a reaction of the type $a + b \rightarrow a + b$.

Let us first note that the element

$$(\mathbf{p}_a;\, q_a\tau_a q_b\tau_b \,|\, W_I \,|\, \mathbf{p}_c;\, q_c\tau_c q_d\tau_d) \tag{7}$$

does not actually depend on $\mathbf{p}_a$ and $\mathbf{p}_c$ separately, but on their mutual orientation [more exactly, on the rotation parameters $g_c g_a^{-1}$; see Eq. (6) and Sec. 2 of Ref. 3]. Let us rotate the pair of vectors $\mathbf{p}_a$ and $\mathbf{p}_c$ together so that $\mathbf{p}_c$ is directed along the original $\mathbf{p}_a$ direction [it is assumed that the whole coordinate system in which the spin state of the beam and target is described is also rotated; so that the spin is expressed in exactly the same way as before the rotation; see Sec. 2 of Ref. 3 and Eq. (6)]. In this procedure (7) does not change, but the rotated position of p_a and its nonrotated position are symmetric with respect to the nonrotated position of p_c. More exactly, if the z axis is chosen along p_a, then (7) is equal to W_I for the case in which the products of the inverse reaction are emitted at the angles ϑ, $\varphi + \pi$, if the element $(-p_c | W_D | -p_a)$ [which can be shown by similar considerations to be equal to $(p_c | W_D | p_a)$] describes the state in which the products of the direct reaction are emitted at the angles ϑ, φ.

Thus, after some obvious changes of notation, (5) can be written

$$(q_a\tau_a q_b\tau_b | W_{\mathrm{I}}(\vartheta, \varphi + \pi) | q_c\tau_c q_d\tau_d) = (-1)^{q_a+q_b+q_c+q_d+\tau_a+\tau_b+\tau_c+\tau_d} (q_c, -\tau_c, q_d, -\tau_d | W_{\mathrm{D}}(\vartheta, \varphi) | q_a, -\tau_a, q_b, -\tau_b). \tag{8}$$

3. In this section we shall show how to use (8) to obtain several relations between the experimental observables from the direct and inverse reactions. Generally, speaking, what we shall do is replace the coefficients W of (8) by their expressions in terms of angular distributions, polarization vectors, and other statistical tensors of the direct and inverse reactions, quantities which are given directly by experiment. It is clear that one can obtain as many relations between these quantities as there are coefficients W.

First we shall make one remark (A) and introduce some abbreviated notation (B).

A. Let us go over from the probabilities for processes to their cross sections (adjusted to the conditions of the experiment) and therefore introduce the statistical tensor $\rho'_{\mathrm{rt}}(\mathbf{n}, p; q_1\tau_1 q_2\tau_2)$ of the reaction products renormalized to the current, which has been defined in Appendix II of Ref. 3 [so that $\rho'_{\mathrm{rt}}(\mathbf{n}, p; 0, 0, 0, 0)$ is equal to the differential cross section per unit solid angle $\sigma(\mathbf{n})$].* It has been shown that the transition from ρ' to ρ'_{rt} is accomplished by replacing $N/(4\pi)^2$ by $h^2/4p_a^2$ in (6) and by $h^2/4p_c^2$ in (6′). Therefore if we replace ρ' by ρ'_{rt} in Eqs. (1) and (1′), then W_D and W_I must be replaced by

$$(h^2/4p_a^2)(4\pi)^2 W_{\mathrm{D}}/N \text{ and } (h^2/4p_c^2)(4\pi)^2 W_{\mathrm{I}}/N.$$

B. Let us denote the statistical tensors of particles a and b in the direct reaction

$$-\rho(\mathbf{p}_a; 0, 0, 0, 0), \rho(\mathbf{p}_a; 1, -1, 0, 0), \rho(\mathbf{p}_a; 1, 0, 0, 0), \rho(\mathbf{p}_a; 1, 1, 0, 0),$$

$$\rho(\mathbf{p}_a; 0, 0, 1, -1), \rho(\mathbf{p}_a; 0, 0, 1, 0), \ldots \rho(\mathbf{p}_a; 2, -2, 0, 0), \ldots \rho(\mathbf{p}_a; 2i_a, 2i_a, 0, 0),$$

$$\ldots \rho(\mathbf{p}_a; 0, 0, 2i_b, 2i_b), \rho(\mathbf{p}_a; 1, -1, 1, -1)^{\dagger}, \ldots \rho(\mathbf{p}_a; 2i_a, 2i_a, 2i_b, 2i_b),$$

by x_0 (which is equal to unity, as indicated in the preceding footnote *), x_1, x_2, x_3, ..., x_m [where $m = (2i_a + 1)^2(2i_b + 1)^2 - 1$]. In this way the components of all the statistical tensors are renumbered by a single index. We shall use x'_0, x'_1, ..., x'_n and y'_0 (= 1), y'_1, ..., y'_n [where $n = (2i_c + 1)^2 \times (2i_d + 1)^2 - 1$] for particles c and d in the direct and inverse reactions, respectively, and y_0, y_1, ..., y_m for the statistical tensors of particles a and b in the inverse reaction. The letter a with new indices shall be used to denote the products of $(4\pi h)^2/4N$ [see remark (A)] with both the elements W_D and the equal W_I corresponding to them.

Bearing in mind remark (A), Eq. (1) for the direct reaction and the corresponding equation (1′) for the inverse reaction take on the form

$$\begin{aligned} x'_0 &= Fp_a^{-2}(a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0m}x_m), \\ x'_1 &= Fp_a^{-2}(a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x'_n &= Fp_a^{-2}(a_{no}x_0 + \cdots + a_{nm}x_m), \end{aligned} \tag{9}$$

in the new notation, where

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*We note that the statistical tensor $\rho(\mathbf{n}, p; 0, 0, 0, 0)$ of the beam and target is simply equal to unity (normalization of the density matrix of the initial state).

†All the statistical tensors following on this one are products of pairs of preceding one [divided by $\rho(\mathbf{p}; 0, 0, 0, 0)$; see Appendix I in Ref. 3].

$$F = [(2i_c + 1)(2i_d + 1)]^{1/2} [(2i_a + 1)(2i_b + 1)]^{-1/2};$$

$$\begin{aligned} y_0 &= F^{-1} p_c^{-2} (a_{00} y_0' + a_{30} y_1' - a_{20} y_2' + a_{10} y_3' + \cdots), && \text{n terms} \\ y_1 &= F^{-1} p_c^{-2} (a_{03} y_0' + a_{33} y_1' - a_{23} y_2' + \cdots), && \text{"} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= F^{-1} p_c^{-2} (a_{0,m-4i_a-4i_b} y_0' + \cdots). && \text{"} \end{aligned} \tag{10}$$

The rectangular matrix A_I of the a_{ik} in (10) is obtained by transposing the matrix A_D of the coefficient a_{ik} in (9) and then performing certain definite permutations and sign changes for some of the elements. For instance, the (1, 2) element (second row, third column) of A_I is

$$(1, -1, 0, 0 \,|\, W_I(\vartheta, \varphi + \pi) \,|\, 1, 0, 0, 0)(2\pi h)^2/N = (-1)(1, 0, 0, 0) \,|\, W_D(\vartheta, \varphi) \,|\, 1, 1, 0, 0)(2\pi h)^2/N = -a_{23}.$$

The method for completely studying the reaction, that is for finding experimentally all the W (or a_{ik})* can be described as follows.

Let us assume we are considering the reaction a + b → c + d at some given energy. One must perform m experiments with the beam and target polarized in different ways (see below), measuring all the statistical tensors of the reaction products in each of these experiments (for all angles of emission). Then, for instance, a_{00}, a_{01}, ..., a_{0m} are found from the set of equations (note that all the $x_0^{(k)}$ are equal to unity; see the footnote * on page 751)

$$\begin{aligned} (x_0')^{(1)} &= F p_a^{-2} (a_{00} x_0^{(1)} + a_{01} x_1^{(1)} + \ldots + a_{0m} x_m^{(1)}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (x_0')^{(m)} &= F p_a^{-2} (a_{00} x_0^{(m)} + a_{01} x_1^{(m)} + \ldots + a_{0m} x_m^{(m)}), \end{aligned} \tag{11_0}$$

if the determinant $\| x_i^{(k)} \|$ does not vanish (this is the condition that the spin states of the beam and target are different in our m experiments).

In principle, the solution of the problem posed in this section can now be given as follows: one must find expressions for the a_{ik} in terms of the statistical tensors of the direct reaction [by making a complete study of the direct reaction and solving a set of equations (11_i)] and setting them equal to the expressions for the a_{ik} in terms of the statistical tensors of the inverse reaction (found in the same way). Furthermore, if all the a_{ik} are known, then by inserting them into (10) we immediately obtain m relations between the statistical tensors of the direct and inverse reactions.

4. The well known relation of "semidetailed balance"

$$p_a^2 (2i_a + 1)(2i_b + 1) \sigma_D(\vartheta) = p_c^2 (2i_c + 1)(2i_d + 1) \sigma_0(\vartheta)$$

is a simple consequence of (8). We present still another illustration of the results of Sec. 3 (see also the Introduction).

Suppose we are interested in the angular distribution $\sigma_D'(\vartheta, \varphi)$ of the products of the direct reaction in which the beam a and target b are polarized in an arbitrary way. This is given by the first relation of (9). The a_{ik} can be found by measuring all the statistical tensors y_1, y_2, ..., y_m for the inverse reaction with completely unpolarized beam and target. If the latter are known, one can find the difference between the desired $\sigma_D'(\vartheta, \varphi)$ and the angular distribution $\sigma_D(\vartheta, \varphi)$ when a and b are unpolarized, namely

$$\sigma_D'(\vartheta, \varphi) - \sigma_D(\vartheta, \varphi) = F^2 p_c^2 p_a^{-2} (y_3 x_1 - y_2 x_2 + y_1 x_3 + y_6 x_4 - \ldots). \tag{12}$$

In particular, assume that only the target b is polarized so that only its polarization vector (whose cyclic† components x_1, x_2, and x_3 we shall denote by ρ_{-1}, ρ_0, and ρ_{+1}) is nonzero. Then in the inverse

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*If all the W are known, one must first solve the set of linear equations (6) for the products

$$(s_1' l_1' \,|\, R^{J_1 E} \,|\, s_1 l_1)(s_2' l_2' \,|\, R^{J_2 E} \,|\, s_2 l_2)^*,$$

in order to find the elements of the R matrix, after which the unique values of these products (though not yet of the elements themselves) will be found.

†The relations between the cyclic and the ordinary cartesian components of a vector are

$$\rho_{-1} = (\rho_x + i\rho_y)\sqrt{2}; \quad \rho_0 = \rho_z; \quad \rho_{+1} = (-\rho_x + i\rho_y)/\sqrt{2}.$$

reaction it is sufficient to measure only the components ρ'_{-1}, ρ'_0, and ρ'_{+1} of the statistical vector of particles b. Equation (12) then becomes

$$\sigma'_{\mathrm{D}}(\vartheta, \varphi) - \sigma_{\mathrm{D}}(\vartheta, \varphi) = F^2 p_c^2 p_a^{-2} (\rho'_{+1}\rho_{-1} - \rho'_0 \rho_0 + \rho'_{-1}\rho'_{+1}) = -F^2 p_c^2 p_a^{-2} (\rho'(\vartheta, \varphi + \pi)\rho). \qquad (13)$$

Since the beam and target are unpolarized in (10), it can be shown that the statistical vectors of particles b emitted at angles ϑ, φ and ϑ, $\varphi + \pi$ are of equal magnitude and oppositely directed (and both perpendicular to the reaction plane). In place of ρ' let us introduce the polarization vector $\mathbf{N}'$ (which for particles of spin $\frac{1}{2}$ is equal to twice the expectation value of the vector spin operator), so that $\rho'(\vartheta, \varphi) = \mathbf{N}'(\vartheta, \varphi)\,\sigma_{\mathrm{I}}(\vartheta, \varphi)$ (where $\sigma_{\mathrm{I}}(\vartheta, \varphi)$ is the cross section for the inverse reaction with unpolarized c and d). Rewriting (13) (we note that $\rho = \mathbf{N}\cdot 1$) in the form

$$\sigma'_{\mathrm{D}}(\vartheta, \varphi) - \sigma_{\mathrm{D}}(\vartheta, \varphi) = F^2 p_c^2 p_a^{-2} (\mathrm{N}'(\vartheta, \varphi)\rho)\,\sigma_{\mathrm{I}}(\vartheta, \varphi), \qquad (14)$$

we obtain a relation for arbitrary spins which has often been obtained in the literature for various special cases (see, for instance, Lapidus[7] and the literature he refers to).

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ON THE METHOD OF "SECOND QUANTIZATION" IN PHASE SPACE

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In order to give a statistical description of processes in systems of interacting particles we use as independent variables the numbers of particles in different points of coordinate-momentum phase space, which at every point of phase space are random functions of the time. We give classical and quantum mechanical equations for these functions; from these equations we can, by averaging the random functions over their distributions, obtain a set of equations for the classical and quantum mechanical distribution functions. To illustrate the possible use of this method we obtain expressions for the excitation spectrum and correlation function for systems of particles interacting through central forces in the case where close interactions are unimportant. This method is convenient for considering system of particles and fields, in particular, electromagnetic interactions. It is possible to generalize this method for the relativistic case.

IN the present paper we give a method to introduce collective variables to describe a system of interacting particles which is slightly different from the considerations in the papers of Tomonaga,[1] Bogoliubov and Zubarev[2,3], and Bohm and Pines.[4] We introduce as independent variables the numbers of particles in different points of coordinate-momentum phase space. In this sense this method is analogous to the method of second quantization and we have therefore called it in Ref. 5 the method of second quantization in phase space.* The method under consideration is convenient because it makes it possible in a number of cases to describe in a unified and fairly simple way processes which occur both in classical and quantum mechanical systems. We give in the present paper, on the basis of this representation, conclusions derived from a set of equations for classical[6] and quantum mechanical distribution functions.[7] It is shown how this method can be used to investigate the excitation spectrum and correlation function of systems of interacting particles.

We consider a classical system of N particles interacting through central forces. We denote by

$$N_{\mathbf{qp}}(t)d\mathbf{q}\,d\mathbf{p} = \sum_{1\leqslant i\leqslant N} \delta(\mathbf{q}-\mathbf{q}_i)\,\delta(\mathbf{p}-\mathbf{p}_i)\,d\mathbf{q}\,d\mathbf{p}$$

the number of particles in a volume element of phase space $d\mathbf{q}\,d\mathbf{p}$. The Hamiltonian function of the system under consideration can in the classical case be written in the following form,

$$H = \sum_{1\leqslant i\leqslant N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{1\leqslant i\neq j\leqslant N} \Phi(|\mathbf{q}_i - \mathbf{q}_j|). \tag{1}$$

Using the definition of the quantities $N_{\mathbf{qp}}(t)$ we can write expression (1) in the following different, equivalent form,

$$H = \int \frac{\mathbf{p}^2}{2m} N_{\mathbf{qp}} d\mathbf{q}\,d\mathbf{p} + \frac{1}{2}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\{N_{\mathbf{qp}}N_{\mathbf{q}'\mathbf{p}'} - \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,N_{\mathbf{qp}}\}\,d\mathbf{q}\,d\mathbf{q}'\,d\mathbf{p}\,d\mathbf{p}'. \tag{2}$$

It is convenient to choose as the initial Hamiltonian function a function which includes the terms $\Phi(\mathbf{q}_i - \mathbf{q}_j)$ for $i = j$. In that case we obtain instead of expression (2) the following expression for the Hamiltonian function

$$H = \int \frac{\mathbf{p}^2}{2m} N_{\mathbf{qp}} d\mathbf{q}\,d\mathbf{p} + \frac{1}{2}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\, N_{\mathbf{qp}}N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}\,d\mathbf{q}'\,d\mathbf{p}\,d\mathbf{p}'. \tag{3}$$

writing the Hamiltonian function in the form (3) presupposes that all particles are identical.

The functions $N_{\mathbf{qp}}(t)$ are for all fixed values of $\mathbf{q}$ and $\mathbf{p}$ random functions of the time. We denote by $F(\ldots N_{\mathbf{qp}} \ldots)$ the multidimensional distribution law for the different values of the $N_{\mathbf{qp}}(t)$. In the following we shall define averages using the function $F(\ldots N_{\mathbf{qp}} \ldots)$.

Using expression (3) for the Hamiltonian function we fine the equations of motion in the representation under consideration,

$$\dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{p}}\frac{\delta H}{\delta N_{\mathbf{qp}}} = \frac{\mathbf{p}}{m}, \qquad \dot{\mathbf{p}} = -\frac{\partial}{\partial \mathbf{q}}\frac{\delta H}{\delta N_{\mathbf{qp}}} = -\frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\,N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'. \tag{4}$$

Equations (4) combined with the equation of continuity in phase space $dN_{\mathbf{qp}}(t)/dt = 0$ leads to the following equation for $N_{\mathbf{qp}}(t)$,

$$\frac{\partial N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\,N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'\,\frac{\partial N_{\mathbf{qp}}}{\partial \mathbf{p}} = 0. \tag{5}$$

Equation (5) has the same form as the classical self-consistent field equation which was investigated in detail by Vlasov[9] and Landau.[10] However, there is an essential difference between the two equations. Indeed, the classical self-consistent field equation is an approximate equation for distribution functions which can be obtained from equations for the distribution function of the whole system $f_N(\mathbf{q}_1 \ldots, \mathbf{p}_1 \ldots, t)$ assuming this function to be factorized

$$f_N = \prod_{1\leqslant i\leqslant N} f_1(\mathbf{q}_i, \mathbf{p}_i, t),$$

or as a first approximation in an expansion in powers of a small parameter.[11] The self-consistent field equation does not take into account the correlation produced by the interaction between the particles.

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*M. Schönberg[8] has published under an analogous title several interesting papers, but his aim and results are different and we shall therefore not discuss the results of these papers.

Equation (5), however, is an equation for the random functions $N_{qp}(t)$. As will be seen presently we can, by averaging by means of the phase-space distribution function $F(\dots N_{qp}\dots)$, obtain from Eq. (5) a set of equations for the distribution functions $f_1(q, p, t)$, $f_2(q, p, q', p', t)$, which are considered in the well-known monograph of Bogoliubov's[6] and we can therefore consider correlations in every approximation. The relation between Eq. (5) and the self-consistent field equations for the distribution functions is analogous to the relation between the equations for the state function Ψ in the theory of second quantization and the Hartree equations. We must finally note that in Vlasov's papers[9] also another interpretation of the equation for the function $f_1(q, p, t)$ was suggested.

We shall now establish the connection between the average of the product of functions $N_{qp}(t)$ in different points of phase space and the functions f_1, f_2. By comparison with the expressions for the average values of functions of dynamical variables which are additive, binary, etc. averages, obtained by using the distribution functions f_1, f_2, and the function F, one can obtain the relation between the distribution functions and the average values of the products of the $N_{qp}(t)$. In the classical case these relations have the following form,

$$Nf_1(\mathbf{q}, \mathbf{p}, t) = \overline{N}_{\mathbf{qp}}(t); \tag{6}$$

$$N(N-1)f_2(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) = \overline{N_{\mathbf{qp}}N_{\mathbf{q}'\mathbf{p}'}} - \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,\overline{N}_{\mathbf{qp}}; \tag{7}$$

$$N(N-1)(N-2)f_3(\mathbf{q}, \mathbf{q}', \mathbf{q}'', \mathbf{p}, \mathbf{p}', \mathbf{p}'', t)$$
$$= \overline{N_{\mathbf{qp}}(t)\,(N_{\mathbf{q}'\mathbf{p}'}(t) - \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}'))\,(N_{\mathbf{q}''\mathbf{p}''}(t)} - \delta(\mathbf{q}'-\mathbf{q}'')\,\delta(\mathbf{p}'-\mathbf{p}'') - \delta(\mathbf{q}-\mathbf{q}'')\,\delta(\mathbf{p}-\mathbf{p}'')). \tag{8}$$

From Eqs. (6) to (8) one can easily find the inverse expressions of the average values of the products of the $N_{qp}(t)$ in terms of the functions f_1, f_2. The functions $f_i(q_1 \dots, p_1 \dots)$ are normalized as follows,

$$\int f_i\,(d\mathbf{q}_i)\,(d\mathbf{p}_i) = 1.$$

In Eqs. (6) to (8) one can take the limit $N \to \infty$. In that case it is convenient to use instead of the functions N_{qp} quantities referring to unit volume or to one particle. If we consider the quantities $N_{qp}(t)$ at different points in time, we obtain the many-time distribution functions which were considered by van Hove[12] and Tolmachev.[13]

Using Eq. (5) and expressions (6) to (8) we can obtain a set of equations for the usual distribution functions $f_1, f_2, \dots$ and for the many-time distribution functions $f_1(q, p, t)$, $f_2(q, p, q', p', t, t')$, ... Indeed, averaging (5) and using (6) and (7) we get the first equation of the set which connects the functions f_1 and f_2,

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f_1}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\left\{(N-1)\frac{\partial}{\partial \mathbf{p}}f_2(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) + \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\frac{\partial}{\partial \mathbf{p}}f_1\right\}d\mathbf{q}'\,d\mathbf{p}'. \tag{9}$$

The occurrence of an extra term is connected with our choice of Eq. (3) for the Hamiltonian function. To get the next equation of the set one must multiply Eq. (5) by $N_{q'p'}(t')$ and average, and so on.

To obtain the solution of several problems it is convenient to use not Eq. (5), but the equation for the difference between the functions N_{qp} and their average values, that is, for the functions $\delta N_{qp}(t) = N_{qp} - \overline{N_{qp}(t)}$. The mean square deviation is connected with the correlation function,

$$\overline{\delta N_{\mathbf{qp}}(t)\,\delta N_{\mathbf{q}'\mathbf{p}'}(t)} = g(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}', t) + \delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}')\,\overline{N}_{\mathbf{q},}(t). \tag{10}$$

From Eq. (5) we get the following equation for the δN_{qp},

$$\frac{\partial}{\partial t}\delta N_{\mathbf{qp}} + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{q}}\delta N_{\mathbf{qp}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\frac{\partial}{\partial \mathbf{p}}\{\overline{N}_{\mathbf{q}'\mathbf{p}'}\delta N_{\mathbf{qp}} + \overline{N}_{\mathbf{qp}}\delta N_{\mathbf{q}'\mathbf{p}'} + \delta N_{\mathbf{qp}}\delta N_{\mathbf{q}'\mathbf{p}'} - \overline{\delta N_{\mathbf{qp}}\delta N_{\mathbf{q}'\mathbf{p}'}}\}\,d\mathbf{q}'\,d\mathbf{p}' = 0, \tag{11}$$

from which we can easily obtain an equation for the mean square deviation,

$$\frac{\partial}{\partial t}\delta N_{\mathbf{qp}}\overline{\delta N_{\mathbf{q}'\mathbf{p}'}} + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{q}}\delta N_{\mathbf{qp}}\overline{\delta N_{\mathbf{q}'\mathbf{p}'}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)$$
$$\times \frac{\partial}{\partial \mathbf{p}}\{\overline{N}_{\mathbf{qp}}\overline{\delta N_{\mathbf{q}'\mathbf{p}'}\,\delta N_{\mathbf{q}''\mathbf{p}''}} + \overline{N_{\mathbf{q}''\mathbf{p}''}}\,\overline{\delta N_{\mathbf{qp}}\,\delta N_{\mathbf{q}'\mathbf{p}'}} + \overline{\delta N_{\mathbf{qp}}\,\delta N_{\mathbf{q}'\mathbf{p}'}\,\delta N_{\mathbf{q}''\mathbf{p}''}}\}d\mathbf{q}''d\mathbf{p}'' = 0. \tag{12}$$

These equations simplify considerably, if we neglect in (12) triple deviations, which is possible when the deviations are small, and in (11) the term $\delta N_{qp}\delta N_{q'p'} - \overline{\delta N_{qp}\delta N_{q'p'}}$. The neglecting of this term in (11) is possible when the interactions are weak, and when we are interested in phenomena for which close interactions play a minor role. The solutions of (11) and (12) given below are only correct under these assumptions.

We consider one particular, very important case when the average distribution of particles in phase space does not depend on time and is uniform in coordinate space, that is, $\overline{N}_{qp}(t) = \overline{N}_{p}$. This case corresponds to statistical equilibrium and represents a random, uniform, stationary process.[14] The mean square deviation depends in this case only on the time difference $|t - t'| = \tau$ and on the relative position $\mathbf{q} - \mathbf{q}'$ of the particles, i.e.,

$$\overline{\delta N_{\mathbf{qp}}(t)\,\delta N_{\mathbf{q'p'}}(t')} \to M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}'). \tag{13}$$

The function $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$ is connected with the space-time correlation function of the stationary, uniform process under consideration. With these assumptions Eq. (11) has the following form,

$$\frac{\partial \delta N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial \delta N_{\mathbf{qp}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\,\delta N_{\mathbf{q'p'}}\,d\mathbf{q}'\,d\mathbf{p}'\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0, \tag{14}$$

and the equation for $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$ has the form,

$$\frac{\partial M}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial M}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\,M(\tau, \mathbf{q}', \mathbf{p}', \mathbf{p}'')\,d\mathbf{q}'\,d\mathbf{p}''\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0. \tag{15}$$

Let us consider the solution of the equation for M in the case where $\tau = 0$. If we take into account the connection between the function $M(\mathbf{q})$ and the correlation function $g(|\mathbf{q}|)$, which we have normalized according to Ref. 6,

$$g(\mathbf{q}, \mathbf{p}) = M(\mathbf{q}, \mathbf{p})\,|_{\overline{n}^2} - \delta(\mathbf{q})\,\overline{N}_{\mathbf{p}}\,|_{\overline{n}^3};\ \ \overline{n} = \int \overline{N}_{\mathbf{p}}\,d\mathbf{p},$$

we get the following equation for the correlation function

$$\sum_{i=1,\,2}\left\{\frac{\mathbf{p}_i}{m}\frac{\partial g}{\partial \mathbf{q}_i} - \frac{\partial}{\partial \mathbf{q}_i}\int \Phi(|\mathbf{q}-\mathbf{q}'|)\,g(\mathbf{q}', \mathbf{p}')\,d\mathbf{q}'\,d\mathbf{p}'\frac{\partial \overline{N}_{\mathbf{p}}}{\partial p_i}\right\} = \sum_{i=1,\,2}\frac{\partial \Phi(|\mathbf{q}|)}{\partial \mathbf{q}}\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}\frac{1}{\overline{n}}. \tag{16}$$

Solving Eq. (16) we get

$$g(|\mathbf{q}|) = \frac{1}{(2\pi)^3\,\overline{n}}\int \frac{\nu(\mathbf{k})\,F(\mathbf{k})}{1-\nu(\mathbf{k})\,F(\mathbf{k})}\,e^{-i\mathbf{kq}}\,d\mathbf{k}, \tag{17}$$

$$M(|\mathbf{q}|) = \frac{\overline{n}}{(2\pi)^3}\int \frac{1}{1-\nu(\mathbf{k})\,F(\mathbf{k})}\,e^{-i\mathbf{kq}}\,d\mathbf{k}, \tag{18}$$

where

$$F(\mathbf{k}) = \int \mathbf{k}\,\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}}\,\frac{m}{\mathbf{kp}}\,d\mathbf{p}, \quad \nu(k) = \int \Phi(|\mathbf{q}|)\,e^{i\mathbf{kq}}\,d\mathbf{q}. \tag{19}$$

Substituting the Maxwell distribution into (18) we get a result which agrees with the correlation function obtained in Ref. 3. If the interaction between the particles is a Coulomb interaction we get from Eq. (18) the Debye expression

$$g(q) = -(1/4\pi\overline{n}r_d^2 q)\exp(-q/r_d).$$

Let us now find an expression for the average energy in the case under consideration. Substituting $N_{qp} = \overline{N}_p + \delta N_{qp}$ into Eq. (2) for the Hamiltonian function and averaging we get the expression

$$\overline{H} = \frac{3N\varkappa T}{2} + \frac{N^2\nu(0)}{2V} + \frac{V}{2}\int \Phi(|\mathbf{q}|)\,M(|\mathbf{q}|)\,d\mathbf{q}.$$

Let us now consider the solution of the equations for $\delta N_{qp}(t)$ and $M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')$ for the case when $\tau \neq 0$.

The solution of Eq. (14) can be written in the form of an infinite sum of harmonic terms with random amplitudes and phases,

$$\int \delta N_{\mathbf{qp}}(t)\, d\mathbf{p} = (2\pi)^{-4} \int n_{\mathbf{k},\,\omega} e^{i\omega t - i\mathbf{kq}} d\omega\, d\mathbf{k}. \tag{20}$$

Substituting (20) into (14) we find from the condition that the latter has a solution that

$$1 = \nu(\mathbf{k}) \int \frac{\mathbf{k}\partial N_{\mathbf{p}}/\partial \mathbf{p}}{(\mathbf{kp}/m) - \omega}\, d\mathbf{p}, \tag{21}$$

which determines for our case the dependence of the possible values of $\omega_{\mathbf{k}}$ on the wave vector $\mathbf{k}$. Expansion (21) agrees with the expression for the distribution function obtained by Vlasov[9] and Landau[10] for the solution of the linearized self-consistent field equation.

Formula (21) determines the form of the space-time correlation function. This follows immediately from the solution of Eq. (15), if we look for this solution in the form

$$\int M(\tau, \mathbf{q}, \mathbf{p}, \mathbf{p}')\, d\mathbf{p}\, d\mathbf{p}' = (2\pi)^{-4} \int f(\omega, \mathbf{k})\, e^{i\omega t - i\mathbf{kq}} d\omega\, d\mathbf{k}. \tag{22}$$

Substituting expansion (22) into (15) we find that we can find a solution only if (21) is satisfied. Without solving (15) we can find an expression for the function $M(\tau, \mathbf{q})$ for a system of particles with Coulomb interactions by using (21) directly. From expression (21) we can for a given value of $\mathbf{k}$ determine the frequency $\omega_{\mathbf{k}}$ and the logarithmic decrement $\sigma_{\mathbf{k}}$ of the vibrations corresponding to the Fourier coefficients $n_{\mathbf{k}}(t)$ obtained from the expansion

$$\int \delta N_{\mathbf{qp}}(t)\, d\mathbf{p} = (2\pi)^{-3} \int n_{\mathbf{k}}(t)\, e^{-i\mathbf{kq}} d\mathbf{k}. \tag{23}$$

In the case of long wavelengths we can immediately write down the expressions for $\omega_{\mathbf{k}}$ and $\sigma_{\mathbf{k}}$, if we use Landau's result[10]

$$\omega_{\mathbf{k}}^2 = \omega_L^2 + 3r_d^2 k^2; \quad \sigma_{\mathbf{k}} = \omega_L \sqrt{\pi/8}\, (kr_d)^{-3} \exp(-1/2k^2 r_d^2). \tag{24}$$

Since $\sigma_{\mathbf{k}} \ll \omega_{\mathbf{k}}$ the Fourier coefficients $n_{\mathbf{k}}(t)$ satisfy the differential equation

$$\ddot{n}_k + 2\sigma_k n_k + \omega_{\mathbf{k}}^2 n_{\mathbf{k}} = 0. \tag{25}$$

From Eq. (25) we find an expression for the spectral density of the correlation function $M(\tau, \mathbf{q})$

$$f(\omega, \mathbf{k}) = A / [(\omega - \omega_{\mathbf{k}})^2 + 4\omega_{\mathbf{k}}^2 \sigma_{\mathbf{k}}^2], \tag{26}$$

where

$$f(\omega, \mathbf{k}) = \int \overline{n_{\mathbf{k}}(t)\, n_{\mathbf{k}}(t+\tau)}\, e^{-i\omega\tau} d\tau = \int M(\tau, \mathbf{k})\, e^{-i\omega\tau} d\tau.$$

From (26) we get

$$M(\tau, \mathbf{k}) = (A / 4\sigma_{\mathbf{k}} \omega_{\mathbf{k}}^2)\, e^{-\sigma_{\mathbf{k}}|\tau|} \cos \omega_{\mathbf{k}}\tau. \tag{27}$$

In the case of statistical equilibrium we can determine the value of A from the condition that for $\tau = 0$ the expression

$$M(\tau, \mathbf{q}) = (2\pi)^{-3} \int M(\tau, \mathbf{k})\, e^{-i\mathbf{kq}} d\mathbf{k}$$

for the correlation function must agree with (19). This condition follows immediately from the definition of the many-time correlation functions.[12,13]

From Eq. (27) it follows that the correlation time is determined by the inverse of the logarithmic decrement $\sigma_{\mathbf{k}}$; this quantity was determined by considering only distant interactions. For small values of $\mathbf{k}$ one can take the close interactions approximately into account if we introduce the average relaxation time between collisions.

Let us now go over to a consideration of the quantum mechanical case.

In the representation of second quantization the Hamiltonian of a system of particles with central force interactions, which corresponds to the Hamiltonian function (3), has the following form,

$$H = -\int \Psi^{+} \frac{\hbar^2}{2m} \Delta_q \Psi dq + \frac{1}{2} \int \Phi(|q - q'|) \Psi^{+}(q) \Psi(q) \Psi^{+}(q') \Psi(q') \, dq \, dq', \tag{28}$$

where Ψ^{+} and Ψ are the quantum mechanical wave functions which satisfy the well known commutation relations. In quantum theory one can define the quantity corresponding to the classical $N_{qp}(t)$ as follows,[5]

$$N_{qp}(t) = (2\pi)^{-3} \int \Psi^{+}(q - \hbar\theta/2) \Psi(q + \hbar\theta/2) e^{-i\theta p} d\theta. \tag{29}$$

We consider first of all the case where we can neglect exchange interactions. By comparison with the expressions for the average values of dynamical quantities which are additive, binary, and so on, we can find relations between the average values of products of the functions $N_{qp}(t)$ and the quantum mechanical distribution functions f_1, f_2, ...,

$$N f_1(q, p, t) = \overline{N}_{qp}(t); \tag{30}$$

$$\begin{gathered} N(N-1) f_2(q, p, q', p', t) = \overline{N_{qp} N_{q'p'}} \\ - \int \delta\left(q + \frac{\hbar\theta}{2} - \left(q' - \frac{\hbar\theta'}{2}\right)\right) \overline{N}\left(\frac{q - (\hbar\theta/2) + q' + (\hbar\theta'/2)}{2}, \eta, t\right) \exp\{i\theta(\eta - p) + i\theta'(\eta - p')\} \, d\theta \, d\theta' \, d\eta. \end{gathered} \tag{31}$$

These expressions correspond to the classical expressions (6) and (7). If we use the definition of the $N_{qp}(t)$ we can rewrite expression (28) for H as follows,

$$H = \int \frac{p^2}{2m} N_{qp} dq \, dp + \frac{1}{2} \int \Phi(|q - q'|) N_{qp} N_{q'p'} dq \, dp \, dq' \, dp'. \tag{28'}$$

One can obtain an equation for $N_{qp}(t)$ in the quantum mechanical case by using, for instance, the equation for the quantum mechanical wave function Ψ and the equation has the following form

$$\frac{\partial N_{qp}}{\partial t} + \frac{p}{m} \frac{\partial N_{qp}}{\partial q} = \frac{i}{(2\pi)^3 \hbar} \frac{i}{\hbar} \int \left[\Phi\left(\left|q - q' - \frac{\hbar\theta}{2}\right|\right) - \Phi\left(\left|q - q' + \frac{\hbar\theta}{2}\right|\right)\right] N_{q\eta} N_{q'p'} e^{i\theta(\eta - p)} d\theta \, d\eta \, dq' \, dp'. \tag{32}$$

This equation goes over into the classical equation (5) as $\hbar \to 0$.

Equation (32) has the same form as the quantum mechanical transport equation for the distribution function with a self-consistent field, considered by Silin and the present author.[7,15] As far as the correspondence between these equations is concerned, we can say approximately the same as was said earlier concerning Eq. (5) for the $N_{qp}(t)$ and the classical self-consistent field equation for the distribution function.

The $N_{qp}(t)$ are now, just as in the classical case, random functions of q, p, and t, and there exists a distribution function $F(\ldots N_{qp} \ldots)$ for them. Multiplying the average of Eq. (31) by this function one can obtain a set of quantum mechanical equations for f_1, f_2, and so on.

As in the classical case we shall assume

$$N_{qp} = \delta N_{qp}(t) + \overline{N}_{qp}(t).$$

The equations for $\delta N_{qp}(t)$ and for the mean square deviation have the following form

$$\begin{gathered} \frac{\partial \delta N_{qp}}{\partial t} + \frac{p}{m} \frac{\partial \delta N_{qp}}{\partial q} = \frac{i}{(2\pi)^3 \hbar} \frac{i}{\hbar} \int \left[\Phi\left(\left|q - q' - \frac{\hbar\theta}{2}\right|\right)\right. \\ \left. - \Phi\left(\left|q - q' + \frac{\hbar\theta}{2}\right|\right)\right] \{\overline{N}_{q\eta} \delta N_{q'p'} + \delta N_{q\eta} \overline{N}_{q'p'} + \delta N_{q\eta} \delta N_{q'p'} \\ - \overline{\delta N_{q\eta} \delta N_{q'p'}}\} e^{i\theta(\eta - p)} d\theta \, d\eta \, dq' \, dp'; \quad \frac{\partial}{\partial t} \overline{\delta N_{qp} \delta N_{q'p'}} + \frac{p}{m} \frac{\partial}{\partial q} \overline{\delta N_{qp} \delta N_{q'p'}} = \frac{i}{\hbar} \int \left[\Phi\left(\left|q - q'' - \frac{\hbar\theta}{2}\right|\right)\right. \\ \left. - \Phi\left(\left|q - q'' + \frac{\hbar\theta}{2}\right|\right)\right] \{\overline{N}_{q\eta} \overline{\delta N_{q'p'} \delta N_{q''p''}} + \overline{\delta N_{q'p'} \delta N_{q\eta}} \, \overline{N}_{q''p''} + \overline{\delta N_{q'p'} \delta N_{q\eta} \delta N_{q''p''}}\} e^{i\theta(\eta - p)} d\theta \, d\eta \, dq'' \, dp''. \end{gathered} \tag{33}$$

In the case when the average values of the $N_{qp}(t)$ do not depend on the time and when there is a uniform distribution in coordinate space, that is, when $\overline{N_{qp}(t)} = \overline{N}_p$, the mean square deviation will again [compare Eq. (13)] depend only on $|t - t'| = \tau$ and on the relative position $q - q' = q$.

If we neglect in (33) triple deviations we get the following equation for the function $M(\tau, q, p, p')$ defined by (13)

$$\frac{\partial M}{\partial t} + \frac{p}{m}\frac{\partial M}{\partial q} = \frac{i}{(2\pi)^3\hbar}\,\frac{i}{\hbar}\int\left[\Phi\left(\left|q - q' - \frac{\hbar\theta}{2}\right) - \Phi\left(\left|q - q' + \frac{\hbar\theta}{2}\right|\right)\right]\overline{N}_\eta M(\tau,\ q',\ p',\ p'')\,e^{i\theta(\eta - p)}d\theta\,d\eta\,dq'\,dp''. \qquad (34)$$

In the linear approximation we find that Eq. (32) for $\delta N_{qp}(t)$ has a solution only provided

$$1 = \nu\left(\frac{|p|}{\hbar}\right)\int\frac{\overline{N}_{p+\hbar k/2} - \overline{N}_{p-\hbar k/2}}{\hbar k p/m - \hbar\omega}\,dp. \qquad (35)$$

Expression (35) connects the energy $\hbar\omega$ with the momentum $\hbar k$ of the elementary excitations of the system under consideration and is the counterpart of the classical equation (21) connecting ω and k. In Ref. 5, we have given for a number of cases the excitation spectrum following from Eq. (35). [In that paper we started from the Hamiltonian function defined by Eq. (2) and not the one defined by Eq. (3) so that the corresponding equation for $N_{qp}(t)$ contained an additional term.]

We can obtain an expression for the space-time correlation function either from the solution of Eq. (34) for $M(\tau, q, p, p')$, or by using (35) directly.

If we want to take exchange effects into account, Eq. (30) becomes more complicated, but also in that case can the Hamiltonian (28) be expressed in terms of the N_{qp},

$$H = \int\frac{p^2}{2m}N_{qp}dq\,dp + \frac{1}{2}\int\left\{\Phi(|q - q'|) \mp (2\pi)^{-3}\nu\left(\frac{|p - p'|}{\hbar}\right)\delta(q - q')\right\}N_{qp}N_{q'p'}dq\,dq'\,dp\,dp'. \qquad (36)$$

From Eq. (36) it follows that the consideration of exchange effects can be reduced to a change in the interaction energy of the particles in the system.

The equation for $N_{qp}(t)$ has in this case the following form,

$$\begin{aligned}\frac{\partial N_{qp}}{\partial t} + \frac{p}{m}\frac{\partial N_{qp}}{\partial q} &= \frac{1}{(2\pi)^3\hbar}\int\left\{\Phi\left(\left|q - q' - \frac{\hbar\theta}{2}\right|\right) - \Phi\left(\left|q - q' + \frac{\hbar\theta}{2}\right|\right)\right\}N_{q\eta}N_{q'p'}e^{i\theta(\eta - p)}dq'\,dp'\,d\theta\,d\eta\\ &\mp\frac{1}{(2\pi)^9\hbar}\int\left\{\nu\left(\left|\eta - p' + \frac{\hbar k}{2}\right|\right)\delta\left(r - q' - \frac{\hbar\theta}{2}\right) - \nu\left(\left|\eta - p' - \frac{\hbar k}{2}\right|\right)\right. \qquad (37)\\ &\left.\times\,\delta\left(r - q' + \frac{\hbar\theta}{2}\right)\right\}N_{r\eta}N_{q'p'}\exp\{i\theta(\eta - p) + ik(r - q)\}\,d\theta\,d\eta\,dk\,dr\,dq'\,dp'.\end{aligned}$$

For systems of particles, obeying Fermi statistics, we can use a simpler equation, provided the momenta of the particles differ little from the momentum at the top of the Fermi distribution,

$$\begin{aligned}&\frac{\partial N_{qp}}{\partial t} + \left(\frac{p}{m} - \frac{1}{(2\pi)^3}\frac{\partial}{\partial p}\int\nu\left(\left|\frac{p - p'}{\hbar}\right|\right)\delta(q - q')\,N_{q'p'}dq'\,dp'\right)\frac{\partial N_{qp}}{\partial q}\\ &- \frac{\partial}{\partial q}\int\left\{\Phi(|q - q'|) - \frac{1}{(2\pi)^3}\nu\left(\left|\frac{p - p'}{\hbar}\right|\right)\delta(q - q')\right\}N_{q'p'}dq'\,dp'\,\frac{\partial N_{qp}}{\partial p} = 0. \qquad (38)\end{aligned}$$

Equation (37) has the same form as the quantum mechanical self-consistent field equation for the distribution function which takes the exchange effects into account. From Eqs. (37) and (38) we can by averaging obtain a set of equations for the quantum mechanical distribution functions $f_1(q, p, t)$, $f_2(q, p, q', p', t)$ which include exchange. The first equation of this set, which is obtained by averaging Eq. (38) has, for instance, the following form

$$\begin{aligned}&\frac{\overline{\partial N_{qp}}}{\partial t} + \overline{\left(\frac{p}{m} - \frac{1}{(2\pi)^3}\frac{\partial}{\partial p}\int\nu\left(\left|\frac{p - p'}{\hbar}\right|\right)\delta(q - q')\,N_{q'p'}dq'\,dp'\right)\frac{\partial N_{qp}}{\partial q}}\\ &- \frac{\partial}{\partial q}\int\left\{\Phi(|q - q'|) - \frac{1}{(2\pi)^3}\nu\left(\left|\frac{p - p'}{\hbar}\right|\right)\delta(q - q')\right\}\overline{N_{q'p'}dq'\,dp'\,\frac{\partial N_{qp}}{\partial p}}. \qquad (39)\end{aligned}$$

An approximate solution of the first two equations of this set in the case of a short range force $[\Phi(|q - q'|) = \nu(0)\,\delta(q - q')]$ enables us to obtain the transport equation used by Landau in his theory of Fermi-liquids.[17]

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NUCLEAR REACTIONS IN SUPER-DENSE COLD HYDROGEN

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It is shown that nuclear reactions occurring in cold hydrogen at densities of $10^4 - 10^6$ g/cc via barrier penetration proceed with a probability which is quite noticeable on an astrophysical scale. This fact puts a limit on the possible compression of cold hydrogen, since a celestial body cannot last more than 10^8 years at a density of 0.7 g/cc. Such a density is reached in cold hydrogen under the action of gravitation for a mass close to that of the sun.

IT is known[1-3] that the thermonuclear reactions $p + p = D + e^+ + \tilde{\nu}$, $p + D = He^3 + \gamma$ occur in stars; in addition, at high temperature we have the reaction $He^3 + He^3 = He^4 + p + p$, and a high density the reactions $He^3 + e^- = T + \nu$, $T + p = He^4 + \gamma$.

Schatzman[4] first noted that at high densities and low temperatures deviations from the usual expressions for the rate of thermonuclear reactions should begin to appear, and gave general expressions for the case of a degenerate nuclear gas.

Let us consider reactions occurring in hydrogen compressed to a density of $10^4 - 10^6$ g/cc, at a temperature below 10^6 degrees. Under these conditions hydrogen is a solid, so that the protons are located at the lattice points of a crystal; thermal motion and thermonuclear reactions can be neglected.

It is well known that the Coulomb repulsion of the nuclei, which hinders nuclear reaction, can be overcome by virtue of the quantum-mechanical phenomenon of barrier penetration (tunnel effect). The probability of tunneling is exponentially small, and the exponent depends more strongly on the width than on the height of the barrier.

The separation at which nuclear reaction begins is of order 3×10^{-13}; at this distance the Coulomb energy of two protons is 0.5 Mev. At a hydrogen density of $10^4 - 10^6$ g/cc, the distance between protons is $6.5 \times 10^{-10} - 1.35 \times 10^{-10}$ cm (100 times as great as nuclear distances and only 10 — 50 times smaller than the distance between the nuclei in a hydrogen molecule). At this distance, the Coulomb energy of two protons is 400 — 2000 ev.

Thus a compression to such densities hardly lowers the height of the barrier; however, the wide low energy region, which is most difficult to overcome, is gotten rid of through the internal pressure, and the rate of the nuclear reaction $p + p = D + e^+ + \tilde{\nu}$ turns out to be entirely perceptible on an astronomical scale. To calculate the reaction rate at zero temperature and a given density, we find the equilibrium distance r_0 between nuclei under conditions of close packing. We approximate the dependence of the energy on distance between nuclei by the expression $e^2/r + e^2/(2r_0 - r)$. The initial energy is $2e^2/r_0 + E_0$, where $E_0 = \hbar\omega/2 = e\hbar/\sqrt{Mr_0^3}$ is the zero point vibration energy, and M is the proton mass.

The exponential factor for barrier penetration is

$$B = \exp\{-\varphi(4e/\hbar)\sqrt{2M_r r_0}\},$$

where M_r is the reduced mass, equal to $M/2$, and φ is determined by numerical integration. For small ϵ,

$$\varphi = 0.70 - 0.17\ \varepsilon \ln(1/\varepsilon),\quad \varepsilon = E_0 r_0 / e^2.$$

We determine the probability $\psi^2(0)$, for finding two protons at the same point in the absence of nuclear interaction, approximately, by assuming that we have a diatomic molecule[5] with equilibrium separation r_0 and zero point energy E_0

$$\psi^2(0) = BE_0 M_r^2 e^2/\hbar^4.$$

Finally we determine the reaction rate by using Salpeter's[2] calculations: the reaction probability is $p(\mathrm{sec}^{-1}) = w\psi^2(0)$. The constant w is related to the reaction cross section σ (for the case where the reaction occurs in the laboratory, using a beam of particles), by the formula (for singly-charged ions),

$$\sigma = \frac{w}{v}\frac{2\pi e^2}{\hbar v}\exp\left(-\frac{2\pi e^2}{\hbar v}\right) = w\frac{\pi M e^2}{\hbar}\frac{B}{E} = \frac{S}{E}B,$$

where v is the velocity, E the energy, and S is given in Ref. 2, formula (7). Comparing formulas (9) and (39) of Ref. 2, we find $S = 4 \times 10^{-19}$ barn-ev, so that $w = 5 \times 10^{-40}$ cc/sec. After the reaction of formation of the deuteron, the formation of He^3 should follow extremely rapidly. At the densities we are considering, the maximum energy of electrons E_e (in the degenerate Fermi gas) is greater than the tritium decay energy; therefore the reaction $He^3 + e^- = T + \nu$ also goes rapidly,[2] and after it the reaction $p + T = He^4 + \gamma$. Thus each initial reaction of two protons soon leads to the liberation of ~ 25 Mev of energy. A summary of the results for two densities is given in the table:

ρ (g/cc)	r_0 (cm)	E_0, (ev)	$-\log_{10} B$	p (sec^{-1})	H (erg/g-sec)	E (ev)	P (dyne/cm^2)	T (ev)
$5 \cdot 10^5$	$1.65 \cdot 10^{-10}$	125	8.5	$4 \cdot 10^{-17}$	1000	$2.6 \cdot 10^5$	$5 \cdot 10^{22}$	400
$0.7 \cdot 10^5$	$3.1 \cdot 10^{-10}$	45	11.6	10^{-20}	0.25	$0.7 \cdot 10^5$	$0.2 \cdot 10^{22}$	150

Here H is the rate of energy liberation, P is the pressure of the degenerate electron gas, T is that temperature at which the same reaction rate would be reached through the thermonuclear mechanism.[2]

The considerations presented

above impose a limit on the possible compression of cold hydrogen. In fact, even at a density of 0.7×10^5 (the lower line in the table), with zero initial temperature we will reach a temperature of 150 v after 6×10^7 years, and the speed of the reaction will roughly double, so that a celestial body consisting of cold hydrogen at a density of 0.7×10^5 cannot last more than 10^8 years.

From the well known formula of Emden (cf. Ref. 6), the density of cold hydrogen reaches 0.7×10^5 g/cc at a pressure of 2×10^{21} dyne/cm^2 under the action of gravitation, at the center of a body whose mass is 1.5×10^{33} g, i.e., 0.75 times the mass of the sun.

In 1938, Landau[7] pointed out that at high density a "neutron condensation" occurs. For hydrogen, the limiting energy of the reaction $e^- + p = n + \tilde{\nu}$ is 0.75 Mev. The corresponding density is 10^7 g/cc.

The tunneling reaction sets in much earlier. Consequently, the neutron condensation begins only after the conversion of the hydrogen to helium (or to heavier nuclei), and correspondingly at an even greater density than that considered in Ref. 6.

The reactions p + D, p + T, D + D, D + T can also occur via the cold mechanism and require lower pressures. However, this process will never be of practical importance: the required pressures are so high that, under terrestrial conditions, they can be realized only in a non-stationary state, in an extremely small volume and for extremely short times. For equal expenditure of energy or equal pressure, the tunnel reaction is far inferior to the thermonuclear reaction in heated matter.

I take this opportunity to express my thanks to D. A. Frank-Kamenetskii and A. I. Lebedinskii for interest in the work and for valuable comments.

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BEHAVIOR OF NON-METALLIC CRYSTALS IN STRONG ELECTRIC FIELDS

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An expression is obtained for the number of electron-hole pairs generated in a semiconductor by a uniform electric field. The derivation is made for an arbitrary crystal. The result differs from the usually-employed formula in that it contains an explicit angular dependence and a slightly different dependence on the field. A particularly essential fact is that in the absence of electron-phonon collisions and collisions between electrons themselves the magnitude of the effective potential barrier is determined not by the width of the forbidden band, but by the lower edge of optical absorption (the internal photoeffect), which, as a rule, is considerably greater. This circumstance should lead to an essential increase in the critical fields.

THE presence of a strong electrical field ($E \sim 10^5$ v/cm), as is known, produces in semiconductors additional carriers, the number of which increases sharply with increasing field. The most probable mechanisms causing this fact are, first, shock ionization and second, direct knocking out of valence electrons by the field in the conduction band. This latter mechanism, analogous in a certain sense to cold electron emission from a metal surface, was first considered by Zener[1] in the quasi-classical approximation, which is natural for such a problem. The best expressions obtainable by this method for the number of electrons n passing into the conduction band per unit volume per unit time, is given apparently in the work by McAfee et al.[2] and has the form

$$n = N\frac{eEd}{2\pi\hbar}\exp\left\{-\frac{\pi}{2e\hbar E}\sqrt{2m^*}\Delta^{3/2}\right\}, \tag{1}$$

where N is the number of valence electrons per unit volume, d the crystal lattice period, m^* the effective mass of the electron, Δ the width of the forbidden zone, and e the electron charge.

This formula is somewhat indefinite, since the values of the effective mass of the electron in the valence band and in the conduction band are in general different and it is not clear exactly which value is contained in the exponent. In addition, expression (1) was obtained by solving the unidimensional problem. As will be shown below, the correct allowance for the three-dimensionality leads to certain quite substantial qualitative changes. Finally, a particularly important point, no account is taken in the derivation of Eq. (1) of the scattering of electrons by thermal lattice vibrations, which, as will be shown below, is of decisive significance for this problem.

We shall calculate in this work the probability of the passage of a valence electron into the conduction band by a method already used for this purpose by Houston[3] and representing essentially a method commonly used in perturbation theory to calculate the transition probability per unit time. The entire analysis will be carried out in the so-called single-electron approximation, i.e., the interaction between electrons will be disregarded, with the exception of that portion of the interaction included in the general self-consistent field of the crystal.

The Hamiltonian of the system has in this case the following form:

$$\hat{H} = \hat{H}_{0e} + \hat{H}_{0L} + \hat{H}_{eL} + e\mathbf{Er}; \tag{2}$$

$$\hat{H}_{0e} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 + W(\mathbf{r});\ \hat{H}_{0e}\psi_{0j}(\mathbf{p},\ \mathbf{r}) = \varepsilon_j(\mathbf{p})\psi_{0j}(\mathbf{p},\ \mathbf{r}); \tag{3}$$

$$\hat{H}_{0L} = \frac{1}{2}\sum_{|\mathbf{k}|\leqslant \mathbf{k}_m} \hbar\omega_{\mathbf{k}}(b_{\mathbf{k}}^{+}b_{\mathbf{k}} + b_{\mathbf{k}}b_{\mathbf{k}}^{+}); \tag{4}$$

$$\hat{H}_{eL} = i\sqrt{\frac{\nu}{VN}} \sum_{|\mathbf{k}|<k_m} \alpha(\mathbf{k})\{b_{\mathbf{k}}^{+}e^{-i\mathbf{kr}} - b_{\mathbf{k}}e^{i\mathbf{kr}}\}. \tag{5}$$

In these formulas $\mathbf{r}$ is the electron radius vector, $W(\mathbf{r})$ is the periodic potential of the crystal, $\mathbf{p}$ and $\mathbf{k}$ are the electron quasi-momentum and the phonon wave vector, $\omega_{\mathbf{k}}$ and $\epsilon_j(\mathbf{p})$ are functions that determine the dependence of the phonon frequency on the wave vector and of the electron energy on the quasi-momentum in the band with index j, $b_{\mathbf{k}}^{+}$ and $b_{\mathbf{k}}$ are the creation and annihilation operators of a phonon with wave vector $\mathbf{k}$, V is the normalizing volume, ν the number of valence bands, and finally $\psi_{0j}(\mathbf{p}, \mathbf{r})$ the electron wave function in the following Bloch form

$$\psi_{0j}(\mathbf{p}, \mathbf{r}) = \exp(i\mathbf{pr}/\hbar)\,u_j(\mathbf{p}, \mathbf{r}), \tag{6}$$

where $u_j(\mathbf{p}, \mathbf{r})$ is the periodic solution of the equation

$$-\frac{\hbar^2}{2m}\nabla^2 u_j(\mathbf{p}, \mathbf{r}) - \frac{i\hbar}{m}\mathbf{p}\nabla u_j(\mathbf{p}, \mathbf{r}) + \left\{W(\mathbf{r}) + \frac{p^2}{2m} - \varepsilon_j(\mathbf{p})\right\}u_j(\mathbf{p}, \mathbf{r}) = 0. \tag{7}$$

The interaction Hamiltonian (5) should generally speaking also contain functions of the Bloch type instead of the plane waves. However, in the calculation of the matrix elements for transitions between states (6) allowance for this circumstance gives only a corrective factor on the order of unity, which is insignificant, since we do not specify the form of $W(\mathbf{r})$ anyhow, nor do we consequently specify $\psi_{0j}(\mathbf{p}, \mathbf{r})$

No assumptions whatever are made concerning the form of the function $\alpha(\mathbf{k})$, with the exception of the obvious property $\alpha(\mathbf{k}) \sim k^{1/2}$ for small $\mathbf{k}$.

It is well known that the presence of a homogeneous electric field $\mathbf{E}$ leads to a uniform increase in the quasi-momentum of the electron with time in accordance with the law $\mathbf{p} = \mathbf{p}_0 - e\mathbf{E}t$. In other words, if at the instant $t = 0$ the electron is described by a wave function $\psi_{0j}(\mathbf{p}_0, \mathbf{r})$ then in subsequent instants its wave function will have in the zeroth approximation the form

$$\psi_j(\mathbf{p}_0, \mathbf{r}, t) = \exp\left\{-\frac{i}{\hbar}\int_0^t \varepsilon_j(\mathbf{p}_0 - e\mathbf{E}x)\,dx\right\}\psi_{0j}(\mathbf{p}_0 - e\mathbf{E}t, \mathbf{r}). \tag{8}$$

By virtue the periodic dependence (with the period of the reciprocal lattice) of $\epsilon_j(\mathbf{p})$ and $\psi_{0j}(\mathbf{p}, \mathbf{r})$ on the quasi-momentum $\mathbf{p}$, a uniform increase in the latter means that the electron vibrates within the confines of a single band, if the field is directed along one of the principal crystallographic axes (with a period $2\pi\hbar/eEd$ for the case of a simple cubic lattice). If the field is not aligned with any of the reciprocal-lattice vectors, the motion can have a complex aperiodic character, but is still confined to the same band. The exact wave function should, naturally, contain also terms connected with the transitions into other bands, but these terms will obviously be small. It is therefore natural to seek a solution of the Schrödinge equation

$$i\hbar\,\partial\Psi/\partial t = \hat{H}\Psi \tag{9}$$

in the form of a superposition of products of functions of the form (8) (which already include the fundamental effect of the field — uniform acceleration) and of the phonon wave functions

$$\Psi = \sum_{\substack{j,\,\mathbf{p}_0\\ [N_{\mathbf{k}}]}} c_j([N_{\mathbf{k}}], \mathbf{p}_0, t)\exp\left\{e\mathbf{E}\int_0^t \gamma_j(\mathbf{p}_0 - e\mathbf{E}x)\,dx\right\}\psi_j(\mathbf{p}_0, \mathbf{r}, t)\prod_{\mathbf{k}}(b_{\mathbf{k}}^{+})^{N_{\mathbf{k}}}\Phi_0. \tag{10}$$

The symbol $[N_{\mathbf{k}}]$ denotes the set of occupation numbers $N_{\mathbf{k}}$ corresponding to all possible values of $\mathbf{k}$; Φ_0 is the wave function of the lowest energy state of the lattice; summation over p_0 is carried out over all physically-different states, i.e., over the volume of the first Brillouin zone

$$\gamma_j(\mathbf{p}) = \int_{\Omega_0} u_j^*(\mathbf{p}, \mathbf{r})\,\mathrm{grad}_p u_j(\mathbf{p}, \mathbf{r})\,d\tau. \tag{11}$$

The integral in (11) is taken over the volume of the elementary cell. The quantity (11) is pure imaginary since by virtue of the condition

$$\int_{\Omega_0} u_j^*(\mathbf{p}, \mathbf{r})\,u_j(\mathbf{p}, \mathbf{r})\,d\tau = 1 \text{ we have } \operatorname{Re}\gamma_j(\mathbf{p}) = 0.$$

The advisability of separating out the exponential factor in (10) will be seen from the following. Substituting this expansion into (9), we obtain in the usual manner a system of equations for the coefficients:

$$i\hbar \frac{\partial c_j([N_{\mathbf{k}}], \mathbf{p}_0, t)}{\partial t} = i\hbar \sum_{j' \neq j} \frac{\mathbf{E J}_{jj'}(\mathbf{p}_0 - e\mathbf{E}t)}{\varepsilon_j(\mathbf{p}_0 - e\mathbf{E}t) - \varepsilon_{j'}(\mathbf{p}_0 - e\mathbf{E}t)} c_{j'}([N_{\mathbf{k}}], \mathbf{p}_0, t)\, Q^0_{jj'}(\mathbf{p}_0, t) + i\sqrt{\frac{\nu}{VN}} \sum_{j', |\mathbf{k}'| \leqslant k_m}$$

$$\times \{M_{jj'}(\mathbf{p}_0 - e\mathbf{E}t, \hbar\mathbf{k}')\sqrt{N_{\mathbf{k}'}}\, c_{j'}([N_{\mathbf{k}}] - 1_{\overline{\mathbf{k}'}}, \mathbf{p}_0 + \hbar\mathbf{k}', t)\, Q^+_{jj'}(\mathbf{p}_0, \mathbf{k}', t)$$

$$- M_{jj'}(\mathbf{p}_0 - e\mathbf{E}t, -\hbar\mathbf{k}')\sqrt{N_{\mathbf{k}'} + 1}\, c_{j'}([N_{\mathbf{k}}] + 1_{\mathbf{k}'}, \mathbf{p}_0 - \hbar\mathbf{k}', t)\, Q^-_{jj'}(\mathbf{p}_0, \mathbf{k}', t)\}, \tag{12}$$

where

$$M_{jj'}(\mathbf{p}, \hbar\mathbf{k}) = \alpha(\mathbf{k}) \int_{\Omega_0} u^*_j(\mathbf{p}, \mathbf{r})\, u_{j'}(\mathbf{p} + \hbar\mathbf{k}, \mathbf{r})\, d\tau, \quad \mathbf{J}_{jj'}(\mathbf{p}) = \frac{ie\hbar}{2m} \int_{\Omega_0} \{u_{j'}(\mathbf{p}, \mathbf{r}) \nabla u^*_j(\mathbf{p}, \mathbf{r}) - u^*_j(\mathbf{p}, \mathbf{r}) \nabla u_{j'}(\mathbf{p}, \mathbf{r})\}\, d\tau,$$

$$Q^{\pm}_{jj'}(\mathbf{p}, \mathbf{k}, t) = \exp\left\{\frac{i}{\hbar} \int_0^t [\varepsilon_j(\mathbf{p} - e\mathbf{E}x) - \varepsilon_{j'}(\mathbf{p} - e\mathbf{E}x \pm \hbar\mathbf{k}) \pm \hbar\omega_{\mathbf{k}}]\, dx\right.$$

$$\left. - e\mathbf{E} \int_0^t [\gamma_j(\mathbf{p} - e\mathbf{E}x) - \gamma_{j'}(\mathbf{p} - e\mathbf{E}x \pm \hbar\mathbf{k})]\, dx\right\}, \quad Q^0_{jj'}(\mathbf{p}, t) \equiv Q^+_{jj'}(\mathbf{p}, 0, t).$$

In the derivation of (12) use is made of the identity

$$\mathbf{J}_{jj'}(\mathbf{p}) / e\{\varepsilon_j(\mathbf{p}) - \varepsilon_{j'}(\mathbf{p})\} = \int_{\Omega_0} u^*_j(\mathbf{p}, \mathbf{r})\, \mathrm{grad}_{\mathbf{p}} u_{j'}(\mathbf{p}, \mathbf{r})\, d\tau.$$

The diagonal term in the first sum over j' drops out virtue of the presence of an exponential factor in the expansion (10). The symbols $[N_{\mathbf{k}}] \pm 1_{\mathbf{k}'}$ denote that in the set of numbers $[N_{\mathbf{k}}]$ the quantity $N_{\mathbf{k}'}$ is replaced by $N_{\mathbf{k}'} \pm 1$, and all the remaining quantities remain the same.

To obtain an expression analogous to (1) we must discard from the system of equations (12) those terms containing collisions between electrons and phonons. Taking it into account that at $t = 0$

$$c_j(\mathbf{p}, 0) = \delta_{jV}$$

(the indices V and c are necessary in what follows to denote quantities pertaining to the valence and conduction bands respectively), introducing by way of a new variable the vector $\mathbf{p} = \mathbf{p}_0 - e\mathbf{E}t$, and resolving this vector into a component $\mathbf{p}_{\parallel}$ parallel to the field and a vector $\mathbf{p}_{\perp}$ perpendicular to the field, we obtain

$$c_c(\mathbf{p}_0, t) = \int_{p_{0\parallel}}^{p_{0\parallel} - eEt} \frac{\mathbf{E J}_{Vc}(\mathbf{p})}{\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})} \exp\left\{i \int_{p_{0\parallel}}^{p_{\parallel}} [\varepsilon_c(\mathbf{p}') - \varepsilon_V(\mathbf{p}')] \frac{dp'_{\parallel}}{e\hbar E}\right.$$

$$\left. + \int_{p_{0\parallel}}^{p_{\parallel}} \mathbf{n}\gamma_{Vc}(\mathbf{p}')\, dp'_{\parallel}\right\} \frac{dp_{\parallel}}{eE}, \quad \gamma_{Vc}(\mathbf{p}) \equiv \gamma_c(\mathbf{p}) - \gamma_V(\mathbf{p}). \tag{13}$$

Here $\mathbf{n}$ is a unit vector in the direction of the field.

For simplicity and clarity let us first calculate the integral (13) for the simplest case, when the field is directed along one of the principal crystalline axes of a simple cubic lattice with a period d, and then generalize the results to include the case of a lattice of any symmetry and an arbitrarily oriented field.

In this particular case, as indicated above, the motion of the electron in the band is periodic with a period $2\pi\hbar/eEd$. A natural characteristic of the infiltration is therefore the probability of passing through the conduction band during one period

$$D_0(\mathbf{p}_{\perp}) = \left| \int_{-\pi\hbar/d}^{\pi\hbar/d} \frac{\mathbf{n J}_{Vc}(\mathbf{p})}{e[\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})]} \exp\left\{i \int_{p_{0\parallel}}^{p_{\parallel}} \frac{\varepsilon_c(\mathbf{p}') - \varepsilon_V(\mathbf{p}')}{e\hbar E} dp'_{\parallel} + \int_{p_{0\parallel}}^{p_{\parallel}} \mathbf{n}\gamma_{Vc}(\mathbf{p}')\, dp'_{\parallel}\right\} dp_{\parallel} \right|^2. \tag{14}$$

At a fixed value of $\mathbf{p}_{\perp}$ the functions $\epsilon_c(\mathbf{p})$ and $\epsilon_V(\mathbf{p})$ are different branches of the same infinitely-valued analytic function $\epsilon(\mathbf{p})$ of complex variable $p_{\parallel}$, since they represent different roots of the same eigenvalue problem. Since these functions are close to each other on the real axis, there should be located somewhere near the real axis in the complex plane a branch point $p_{\parallel} = q$, in which

$$\varepsilon_c(q) = \varepsilon_V(q).$$

Obviously, q depends on $\mathbf{p}_\perp$. In the vicinity of the point q the branching is into two bands and $\epsilon_c(p_\parallel) - \epsilon_V(p_\parallel) \sim \sqrt{p_\parallel - q}$. The possibility of branching into a larger number of bands,* as well as the multiplicity of the inverse function $p_\parallel(\epsilon)$ in the vicinity of the point q (which corresponds to $\epsilon_c(p_\parallel) - \epsilon_V(p_\parallel) \sim (p_\parallel - q)^{(2n+1)/2}$) would appear only accidently, under particular selection of the potential $W(\mathbf{r})$, and will therefore not be considered below.

By using the general properties of solutions of second-order differential equations it is possible to show (see Supplement) that in the vicinity of the point q the quantity $\mathbf{n}\gamma_{Vc}(p_\parallel)$ behaves as $(p_\parallel - q)^{-1/2}$ and the factor ahead of the exponent in (14) has at the same point a simple pole with a universal value of the residue $i/4$. The latter circumstance makes it possible to calculate the integral in Eq. (14). For this purpose we introduce into (14) a new variable $y = y(p_\parallel)$, satisfying the following conditions: (1) $dy(p_\parallel)/dp_\parallel$ is real and positive for all real $p_\parallel$ and has the same period as the reciprocal lattice; (2) $dy(p_\parallel)/dp_\parallel$ behaves in the vicinity of the point $p_\parallel = q$ as $(p_\parallel - q)^{-1/2}$ and has neither zeros nor singularities lying closer to the real axis than q.

It is easy to check that the integrand in (14) is single-valued in the vicinity of the point $y = y(q)$ and has at this point a pole with residue $i/2$. An example of such a function is the integral

$$\int_0^{p_\parallel} \frac{dp'_\parallel}{\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})}.$$

Shifting the contour of integration in (14) to the upper half of the y plane, we obtain

$$D_0(\mathbf{p}_\perp) = \pi^2 \left| \exp\left\{ \int_0^q \left[i\,\frac{\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})}{e\hbar E} + \mathbf{n}\gamma_{Vc}(\mathbf{p}) \right] dp_\parallel \right\} \right|^2 \qquad (15)$$

with accuracy to terms that are exponentially small compared with fundamental term. Equation (15) discards, in addition, terms connected with the limits of integration, and therefore not increasing with time.†

The probability of passage after l periods is,

$$D_l(\mathbf{p}_\perp) = D_0(\mathbf{p}_\perp) \left| \sum_{r=0}^{l} e^{irs_0} \right|^2 = \frac{\sin^2[(l+1)s_0/2]}{\sin^2(s_0/2)} D_0(\mathbf{p}_\perp),$$

where

$$s_0(\mathbf{p}_\perp) = \int_{-\pi\hbar/d}^{\pi\hbar/d} \left\{ i\,\frac{\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})}{e\hbar E} + \mathbf{n}\gamma_{Vc}(\mathbf{p}) \right\} dp_\parallel$$

is a rapidly oscillating function of the field ($s_0 \sim 1/E$).

However, taking it into account that s_0 is a function of $\mathbf{p}_\perp$ and averaging over the narrow region $\Delta\mathbf{p}_\perp$, we obtain

$$D_l(\mathbf{p}_\perp) = lD_0(\mathbf{p}_\perp). \qquad (16)$$

Equations (14)–(16) were obtained, as already remarked, for a field directed along one of the principal crystallographic axes of a simple cubic lattice, when the motion of the electron has a simple peri-

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*A distinction must be made between the branching of $\epsilon(p_\parallel)$ for fixed $\mathbf{p}_\perp$, considered here, and the frequently-encountered band degeneracy by virtue of the crystal symmetry. In the latter case the equivalent states belonging to the various zones correspond to different directions of the quasi-momentum, i.e., to different $\mathbf{p}_\perp$.

†A formula analogous to (14) was also used by Franz.[4] However, he did not take into account thereafter the presence of a pole in the integrand, and as a result his final results differed strongly from (15) and the equations that follow. These include the value of the factor ahead of the exponent at the point q, which as shown above, is infinite. If this factor is nevertheless replaced by the most sensible value $\sim d/2\pi\hbar$, the resultant values are several orders of magnitude smaller than (15), and contain a factor ahead of the exponent that is dependent on E ($\sim E^{4/3}$).

odic character. In general, however, these cannot be used, since the motion is generally speaking aperiodic and the probability of infiltration changes from period to period. But this circumstance, which is significant in the calculation of the probability of passage of each electron, does not play any role for the total effect when the valence electrons fully fill the band, since the place of the departing electron is taken during the next period by another electron, which will infiltrate with the same probability. Still another difference lies in the fact that the time interval, to which formula (15) pertains, is no longer equal to $2\pi\hbar/eEd$, but is determined by the length, divided by eE, of the segment of the straight line lying within the first Brillouin zone parallel to the field and passing through point $\mathbf{p}_0$, viz.: $T(\mathbf{p}_0) = \Delta p_{\parallel}(\mathbf{p}_0)/eE$. But even this fact is of no significance for the summary effect, because all the electrons having initial states along this segment have an equal probability of passage, and their number is proportional to $\Delta p_{\parallel}(\mathbf{p}_0)/2\pi\hbar$. The number of electrons passing in the conduction band per unit time is proportional to $\Delta p_{\parallel}(\mathbf{p}_0)/T(\mathbf{p}_0)2\pi\hbar = eE/2\pi\hbar$ and is independent of $\Delta p_{\parallel}$, i.e., of the form of the Brillouin band. The equations that will be obtained below are therefore valid for lattices of all symmetries and for all field orientations.

The total number of electrons passing in the conduction band per unit time and per unit volume is given by the expression

$$n = \frac{eE}{2\pi\hbar} \int D_0(\mathbf{p}_\perp) \frac{d^2 p_\perp}{(2\pi\hbar)^2} = \pi^2 \frac{eE}{2\pi\hbar} \int \left| \exp\left\{ \int\limits_0^{q(\mathbf{p}_\perp)} \left[i\,\frac{\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})}{eE\hbar} + \mathbf{n}\gamma_{Vc}(\mathbf{p}) \right] dp_{\parallel} \right\} \right|^2 \frac{d^2 p_\perp}{(2\pi\hbar)^2}. \qquad (17)$$

The principal contribution to the integral (17) is made by the narrow region near that value of $\mathbf{p}_\perp$, which corresponds to the minimum of the function $\epsilon_c(\mathbf{p}) - \epsilon_V(\mathbf{p})$. Actually, for all specified values of $\mathbf{p}_\perp$ this function reaches a certain relative minimum $\epsilon_{\min}(\mathbf{p}_\perp)$ as $p_\parallel$ is varied. The greater $\epsilon_{\min}(\mathbf{p}_\perp)$, the farther is the branch point $q(\mathbf{p}_\perp)$ from the real axis. At values of $\epsilon_{\min}(\mathbf{p}_\perp)$ that are not too large, $q(\mathbf{p}_\perp) \sim \epsilon_{\min}^{1/2}(\mathbf{p}_\perp)$. The exponent therefore contains a large negative quantity, proportional to $\epsilon_{\min}^{3/2}(\mathbf{p}_\perp)$ and consequently, the integrand has a sharp maximum in the region where the function $\epsilon_c(\mathbf{p}) - \epsilon_V(\mathbf{p})$ has an absolute minimum.

In the vicinity of this point we cannot restrict ourselves to an ordinary quadratic expansion, and must use a somewhat more accurate expression, taking into account the presence of a branch-point surface.

$$\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p}) = \varepsilon_0 \left[1 + \sum_{i,k} (p_i - p_{0i})(p_k - p_{0k}) / m_{ik}\varepsilon_0 \right]^{1/2}. \qquad (18)$$

On the other hand, the quantity

$$\int\limits_0^{q(\mathbf{p}_\perp)} \mathbf{n}\gamma_{Vc}(\mathbf{p})\, dp_{\parallel}$$

can be considered constant within this region with a sufficient degree of accuracy, $-i\mathbf{n}\gamma_0 V c \lesssim 1$.

Elementary integration leads then to the following final result:

$$n = 2\pi\nu \left(\frac{eE}{2\pi\hbar} \right)^2 \left(\frac{m_1 m_2 m_3}{m_{\parallel}^3} \frac{m_{\parallel}}{\varepsilon_0} \right)^{1/2} \exp\left\{ -\frac{\pi}{2e\hbar E} \sqrt{m_{\parallel}}\, \varepsilon_0^{3/2} + \mathbf{n}\gamma_{0Vc} \right\}, \qquad (19)$$

where $m_{\parallel}^{-1} = \sum_i [(\cos^2\gamma_i)/m_i]$, m_i^{-1} are the principal values of the tensor m_{ik}^{-1} [see (18)], and γ_i are the angles between the directions of the field and the principal axes of this tensor, which in general do not coincide with the principal crystal axes.

This formula can be presented in a clearer form if one introduces the "average lattice period" d using the equation $d^3 = \Omega_0 = \nu/N$, where Ω_0 is the volume of the elementary cell. Then

$$n = N \frac{\pi}{2} \frac{eEd}{2\pi\hbar} \left(\frac{m_1 m_2 m_3}{m_{\parallel}^3} \right)^{1/2} \frac{eEd}{(\pi\hbar/d)(\varepsilon_0/m_{\parallel})^{1/2}} \exp\left\{ -\frac{\pi}{2e\hbar E} \sqrt{m_{\parallel}}\, \varepsilon_0^{3/2} + \mathbf{n}\gamma_{0Vc} \right\}. \qquad (20)$$

This expression differs from (1) in that the factor in front of the exponent is dependent on the field. The exponential term contains an explicit angular dependence, which can appear also in crystals of cubic symmetry in the presence of degeneracy of the valence band or of the conduction band.

A more substantial difference lies, however, in the meaning that must be ascribed to the quantities ϵ_0 and $m_{\|}$. As can be seen from the above derivation, ϵ_0 is the width of the forbidden band and m_{ik} is the effective mass of the electron (more accurately, the reduced effective mass of the electron and hole) only in that case when the highest states of the valence band and the lowest state of the conduction band correspondi to the same value of the quasi-momentum. In practice this never happens, and consequently ϵ_0, coinciding with the red boundary of light absorption for a given crystal, is always substantially greater than the width of the forbidden band. By virtue of this, the values of the critical fields, corresponding to a noticeable infiltration, should be considerably greater than those usually expected on the basis of Eq. (1).

If, however, there exists some interaction that changes the quasi-momentum of the electron, a transition is possible from the highest state of the valence band to the lowest state of the conduction band.

Such interactions may be collisions between electrons and electrons or between electrons and phonons or impurity atoms. The field dependence of the probability of an infiltration involving these processes is given by the same exponential factor as in (19), except that the width of the forbidden band enters in place of ϵ_0. In view of the very strong dependence of (19) on ϵ_0, the role of these processes can turn out to be decisive.

Let us note also that while formula (17) is quite rigorous, formulas (19) and (20) are obtained under the assumption that ϵ_0 is small compared with the widths of the valence and the conduction bands, and therefore expansion (18) is valid everywhere within the confines of the forbidden band. For the opposite case, corresponding to the appproximation of strongly bound electrons, it is also possible to obtain a relatively simple expression. In this approximation, as is known,[5] it is possible to retain only the first terms in the Fourier expansion

$$\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p}) = I_0\Big[1 + \sum_{\mathbf{g}} \alpha_{\mathbf{g}} e^{i\mathbf{pg}/\hbar}\Big], \qquad (21)$$

where $\mathbf{g}$ are the vectors of the crystal lattice. In this case $I_0 \approx \epsilon_0 \approx \Delta$ and $\alpha_{\mathbf{g}} \ll 1$. Then

$$n \sim N\frac{eEd}{2\pi\hbar}\,\frac{eEd}{I_0}\exp\Big\{-\frac{I_0}{eEd}\Big[q_0 - \sum_{\mathbf{g}}\frac{\alpha_{\mathbf{g}}\hbar}{\mathbf{ng}}\Big(\exp\Big\{\frac{i\mathbf{p}_m - q_0\mathbf{n}}{\hbar}\mathbf{g}\Big\} - \exp\Big\{i\frac{\mathbf{p}_m\mathbf{g}}{\hbar}\Big\}\Big)\Big]\Big\}, \qquad (22)$$

where q_0 is determined by

$$\sum_{\mathbf{g}} \alpha_{\mathbf{g}} \exp\Big\{\frac{i\mathbf{p}_m - q_0\mathbf{n}}{\hbar}\,\mathbf{g}\Big\} + 1 = 0.$$

It is easy to verify that the fundamental term in the exponent is always of the order $-(I_0/eEd)\ln(1/\alpha)$ where α is the ratio of the widths of the allowable and forbidden bands and consequently, the transmission coefficient diminishes somewhat slower with increasing ϵ_0 than called for formula (19). In this approximation, the exponential factor (22) is close to that obtained by Feuer[6] for the unidimensional case.

It must be emphasized that the entire above analysis of the problem of production of electron-hole pair by the electric field starts out with the far-reaching assumptions on which the band theory of solids is based. It is assumed, in particular, that both the electron states (states of the conduction band) and hole states (valence band) can be obtained by solving a certain single-electron problem with a specified periodic potential, and that the electrons and holes created do not interact with each other. As pointed to the author by L. D. Landau, it would be logical to consider the electrons and holes as two different branches of the excitation spectrum of the crystal, without making any further detailed assumption concerning their nature, and also to take it into account that they are created not free, but interact with each other in accordance with the Coulomb law, which may lead to certain changes in the factor in front of the exponent in (19).

It can be shown that such an analysis will lead in practice to the above results if the Coulomb interaction between the electron and hole are disregarded. There are grounds for hoping that allowance for the latter does not change strongly the final derivations, since the value of the Born parameter $e^2/\mu\hbar v$ (where μ is the dielectric constant and v the relative velocity of the electron and hole), a parameter characteristic of this problem, is of the same order of magnitude as the square root of the ratio of the electron and hole bound-state energy to the width of the forbidden band, and is always small.

In conclusion I thank Professor V. L. Ginzburg and Academician L. D. Landau who made many valuable comments when evaluating the results of this work.

SUPPLEMENT

Let us investigate the behavior of the function

$$J_{cV}(\mathbf{p}) / e\,[\varepsilon_c(\mathbf{p}) - \varepsilon_V(\mathbf{p})] \equiv \int_{\Omega_0} u_c^*(\mathbf{p}, \mathbf{r})\,\mathrm{grad}_p u_V(\mathbf{p}, \mathbf{r})\,d\tau. \tag{1S}$$

in the complex plane. For simplicity all further arguments will concern the unidimensional case. Equation (1S), which takes place on the real axis, cannot be extended to the entire complex plane, since its right half contains an essentially non-analytic operation, complex conjugation. We can, however, use the circumstance that on the real axis $u_j^*(p, x) = u_j(-p, x)$ and rewrite (1S) as

$$\frac{J_{cV}(p)}{e\,[\varepsilon_c(p) - \varepsilon_V(p)]} \equiv \int_{\Omega_0} u_c(-p, x)\,\frac{\partial u_V(p, x)}{\partial p}\,dx. \tag{1'S}$$

In such a form it is possible to continue this equation analytically into the region of complex values of p. Analogously, an analytic continuation of the orthogonality and normalization conditions of the functions $u_j(p, x)$ in the p plane leads to the equation

$$\int_{\Omega_0} u_j(-p, x)\,u_{j'}(p, x)\,dx = \delta_{jj'}. \tag{2S}$$

To investigate the properties of (1'S) in the vicinity of the energy branch point q it is advisable to change to a new independent (generally speaking, complex) variable, the energy ϵ. Corresponding to each value of ϵ are two values of p of opposite sign, and consequently two functions $u_j(\pm p(\epsilon), x) = u_j^{\pm}(\epsilon, x)$. The function $p(\epsilon)$ is regular in the vicinity of the point q. The index indicating the number of the zone can be omitted hereinafter, since the states of the different zones correspond to different energies. On the other hand, at the point where the energies of both zones coincide, i.e., at the branch point, the functions $u_c^{\pm}(\epsilon, x)$ and $u_V^{\pm}(\epsilon, x)$ tend to values that differ only by a factor that is independent of x. Actually, were this not so, the initial Schrödinger equation would have at this energy value four linearly-independent solutions, an impossibility for a second-order equation.

If we detour the branch point ϵ_q in the ϵ plane, then by virtue of the uniqueness of the solution of the differential equation with the given boundary condition (i.e., with p specified), we should obtain a function that differs from the original one only by a multiplying factor (which in general depends on ϵ). Consequently, $u^{\pm}(\epsilon, x)$ can be represented in the form $C^{\pm}(\epsilon)\,v^{\pm}(\epsilon, x)$ where the $v^{\pm}(\epsilon, x)$ are unique functions of ϵ in the vicinity of the point ϵ_q. Furthermore, they can be considered regular at this point, for if they had, for example, a pole of the mth order, the factor $(\epsilon - \epsilon_q)^{-m}$ could be taken out and included in $C^{\pm}(\epsilon)$.

Let us denote by ϵ_1 and ϵ_2 the two values of energy, corresponding to the same value of the quasimomentum. As p approaches q, both these quantities tend to ϵ_q, and then, to accuracy within terms of higher order, we have $\epsilon_2 - \epsilon_q = \epsilon_q - \epsilon_1$. Then

$$v^{\pm}(\varepsilon_2, x) = v^{\pm}(\varepsilon_1, x) + 2(\varepsilon_q - \varepsilon_1)\,\partial v^{\pm}(\varepsilon_1, x)/\partial\varepsilon + \ldots. \tag{3S}$$

Let us insert this expansion into the normalization condition for $v^{\pm}(\epsilon_2, x)$ and take account at the same time of the orthogonality of $v^{+}(\epsilon_2, x)$ and $v^{-}(\epsilon_1, x)$

$$1 = C^+(\varepsilon_2)\,C^-(\varepsilon_2) \int_{\Omega_0} v^-(\varepsilon_2, x)\,v^+(\varepsilon_2, x)\,dx = C^+(\varepsilon_2)\,C^-(\varepsilon_2)\,2(\varepsilon_q - \varepsilon_1) \int_{\Omega_0} v^-(\varepsilon_2, x)\,\frac{\partial v^+(\varepsilon_1, x)}{\partial\varepsilon}\,dx + \ldots. \tag{4S}$$

Let us now compare the right half of (4S) with the quantity of interest to us

$$\int_{\Omega_0} u_c(-p, x)\,\frac{\partial u_V(p, x)}{\partial p}\,dx = C^-(\varepsilon_2)\,C^+(\varepsilon_1)\,\frac{d\varepsilon_1}{dp} \int_{\Omega_0} v^-(\varepsilon_2, x)\,\frac{\partial v^+(\varepsilon_1, x)}{\partial\varepsilon}\,dx. \tag{5S}$$

As a result we obtain the relation

$$\int_{\Omega_0} u_c(-p, x)\,\frac{\partial u_V(p, x)}{\partial p}\,dx = \frac{d\varepsilon_V(p)}{dp}\,\frac{C^+(\varepsilon_V)}{C^+(\varepsilon_c)}\,\frac{1}{2(\varepsilon_q - \varepsilon_V)} \approx \frac{C^+(\varepsilon_V)}{C^+(\varepsilon_c)}\,\frac{1}{4(p - q)}. \tag{6S}$$

It is obviously always possible to choose the functions so as to obtain on the real axis $C^+(\epsilon) = C^-(\epsilon)$. Then this equality is always retained, and in the vicinity of the branch point it follows from this equality that $C^+(\epsilon) = C^-(\epsilon) \sim (\epsilon - \epsilon_q)^{-1/2}$ and, consequently

$$\lim_{p \to q} \frac{C^+(\varepsilon_V)}{C^+(\varepsilon_c)} = \pm i.$$

At first glance it may appear that this deduction is not unique, since $C^\pm(\epsilon)$ are determined on the real axis with an accuracy to within an arbitrary phase factor. However, strictly speaking, we are interested not in the quantity (1S), but in its products by the factor

$$\exp\left\{\int_0^p \gamma_{Vc}(p')\,dp'\right\},$$

which is contained in the intergrand of (14) and which, as can be readily verified from the definition of $\gamma_{Vc}(p)$, is in general independent of this phase factor.

We thus have near the branch point

$$\int_{\Omega_0} u_c(-p, x)\frac{\partial u_V(p, x)}{\partial p}\,dx = \pm\frac{i}{4(p-q)} + \dots \qquad (7S)$$

From the definition of the quantity $\gamma_{Vc}(p)$ it is seen that the expansion in powers of $\epsilon - \epsilon_q$ contains no even terms, and therefore the expansion begins with the term $(\epsilon - \epsilon_q)^{-1} \sim (p - q)^{-1/2}$.

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ELECTION RULES FOR ELECTROMAGNETIC TRANSITIONS IN DEFORMED NUCLEI

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Selection rules for electromagnetic transitions have been obtained for the following asymptotic quantum numbers of deformed nuclei: Ω, Λ, and Σ (the projections of the total, the orbital, and the spin angular momentum respectively of a particle on the axis of an elongated nucleus), n_z (the oscillator quantum number along the axis), and N (the principal oscillator quantum number.). Taking into account these selection rules allows an explanation of the discrepancies between theoretical and experimental values of transition probabilities. A first order forbiddenness in one of the asymptotic quantum numbers decreases the transition probability by a factor 10 — 100.

1. INTRODUCTION

LET a nuclear state be characterized by the energy E_i, the angular momentum I_i, its projection on an axis fixed in space M_i, and the parity π_i. It will be able to make a transition to a state with E_f, I_f, M_f, and π_f with emission of a γ-quantum of energy $\hbar\omega = E_i - E_f$ and multipolarity λ. The probability of such a transition is given[1] by

$$T(\alpha\lambda, I_i \rightarrow I_f) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2\lambda+1} B(\lambda), \tag{1}$$

where $B(\lambda)$ is the reduced transition probability

$$B(\lambda) = \sum_{\mu,\, M_f} |\Psi_f^* \mathfrak{M}(\alpha\lambda, \mu)\, \Psi_i\, d\tau|^2, \tag{1a}$$

and $\mathfrak{M}(\alpha\lambda, \mu)$ — the operator of the multipole moment. The α denotes the electric or magnetic character of the transition.

For a multipole transition to be possible the following selection rules must be fulfilled

$$|I_i - I_f| \leqslant \lambda \leqslant I_i + I_f, \tag{2}$$

$$M_f - M_i = \mu, \tag{2a}$$

$$\pi_i\pi_f = \begin{cases} (-1)^{\lambda} & \text{for } E\lambda, \\ (-1)^{\lambda+1} & \text{for } M\lambda. \end{cases} \tag{3}$$

The dependence of the transition probability on the nuclear structure is contained in (1) in the matrix element of the multipole operator. One consequently has to use in actual computations a nuclear model of one kind or another. Because of its simplicity one usually takes the shell model. The emission of γ-rays is then treated as a single-body process. Utilizing the known expressions for the multipole-moment operators, one can derive formulae[1] for the probability of emission of electromagnetic radiation of multipolarity λ. In the region of validity of the shell model the agreement between experiment and theory is in general not bad. The ratio between the observed and the calculated transition probability (denoted in the following by F) is close to unity. However, in the region of highly deformed nuclei ($150 < A < 190$ and $222 < A < 256$) there is a strong disagreement between the experiment and the values calculated by the shell model. This is not unexpected since in this region the unified model of Bohr and Mottelson[2,3] applies.

Nilsson and Mottelson[4,5] worked out single particle states in deformed nuclei. They classified the states according to the quantum numbers Ω, Λ, Σ (projections of the total angular momentum, the orbital angular momentum and the spin of the particle respectively on the elongated nuclear axis) n_z, $n_\perp$

(oscillator quantum numbers along the elongated axis and in a plane perpendicular to it respectively) an N (principal oscillator quantum number).

Alaga et al.[6] give an additional selection rule for electromagnetic transitions in deformed nuclei. Let K be the projection of the total (collective and internal) angular momentum on the elongated nuclear axi Then

$$|K_f - K_i| \equiv |k| \leqslant \lambda. \tag*{(4}$$

The degree of K-forbiddenness is characterized by the number ν

$$\nu = |k| - \lambda \tag*{(5}$$

The experimental data show that in strongly deformed nuclei K is a "good" quantum number and $\nu = 1$ decreases the transition probability by a factor of about 100. However, selection rules (2) — (4) ar still insufficient to explain the discrepancies between experiment and theory. It frequently turns out that the F-factor for transitions allowed with respect to these selection rules is very small. In this connection it is of interest to investigate selection rules with respect to the asymptotic quantum numbers and t determine how they are obeyed in deformed nuclei.

2. ASYMPTOTIC SELECTION RULES

We consider the case of strong nucleon-nuclear surface coupling. Here the spin-orbit coupling is destroyed and the projections of the orbital and spin angular momentum on the nuclear axis, Λ and Σ, are separately constants of the motion. Furthermore, the oscillations of the nucleon along the nuclear axis and perpendicularly to it proceed independently. A natural choice here seems the n_z, $n_\perp$, Λ, Σ, takin as eigenfunctions the functions of an anisotropic harmonic oscillator in cylindrical coordinates. Howeve the particle states in deformed nuclei are commonly characterized by the quantum numbers N, n_z, Λ, Ω $(= \Lambda + \Sigma)$ and it is therefore necessary to work in this representation.

We now consider the transition of a single particle from the state I_i, $K_i = \Omega_i$, π_i, N_i, n_{zi}, Λ_i, Σ_i into the state I_f, $K_f = \Omega_f$, π_f, N_f, n_{zf}, Λ_f, Σ_f with the emission of electromagnetic radiation of multipolarity λ, k. We assume that the transition is allowed with respect to (2) — (4) with respect to total angular momentum I, its projection K, and parity π. We write the operator of the electric multipole moment in the form

$$\mathfrak{M}(E\lambda, k) = \left[e + (-1)^\lambda \frac{Z}{A^\lambda}\right]^2 \left(\frac{\hbar}{M\omega_0}\right)^{\lambda/2} r^\lambda Y_{\lambda k}. \tag*{(6}$$

Here the contribution of the particle magnetic moment and of the nuclear collective motion has not been taken into account. The latter can have a considerable influence on E2 transitions. We write for the magnetic multipole moment operator[3,4]

$$\mathfrak{M}(M\lambda, k) = (e\hbar/2Mc)(\hbar/M\omega_0)^{(\lambda-1)/2}\nabla(r^\lambda Y_{\lambda k})[g_s \mathbf{s} + g_l \mathbf{l}]. \tag{7}$$

Here the contribution due to the collective motion has also been omitted. It can be taken into account for M1 transitions by replacing g_s and g_ℓ by $g_s - g_R$ and $g_\ell - g_R$ respectively; $g_R \approx Z/A$.

The selection rules in (7) are essentially given by the terms $\nabla(r^\lambda Y_{\lambda k})\mathbf{s}$ and $\nabla(r^\lambda Y_{\lambda k})\boldsymbol{\ell}$. We rewrite them in the following form

$$\nabla(r^\lambda Y_{\lambda k})\mathbf{s} = \sqrt{(2\lambda+1)/(2\lambda-1)}\, r^{\lambda-1} \left\{-\frac{1}{2}\sqrt{(\lambda+k)(\lambda+k-1)}\, Y_{\lambda-1,\, k-1}(s_x + is_y) \right.$$
$$\left. + \frac{1}{2}\sqrt{(\lambda-k)(\lambda-k-1)}\, Y_{\lambda-1,\, k+1}(s_x - is_y) + \sqrt{\lambda^2 - k^2}\, Y_{\lambda-1,\, k} s_z\right\}. \tag{8}$$

A similar expression for $\nabla(r^\lambda Y_{\lambda k})\boldsymbol{\ell}$ can be obtained by replacing in (8) $(s_x \pm is_y)$ and s_z by $(\ell_x \pm i\ell_y)$ and ℓ_z respectively. The operator $s_x + is_y$ in (8) corresponds to the transition $\Sigma_i = -\frac{1}{2} \rightarrow \Sigma_f = +\frac{1}{2}$; $s_x - is_y$ corresponds to the transition $\Sigma_i = \frac{1}{2} \rightarrow \Sigma_f = -\frac{1}{2}$, and s_z, together with the operator for the orbital magnetic moment $\nabla(r^\lambda Y_{\lambda k})\boldsymbol{\ell}$, corresponds to transitions with $\Sigma_i = \Sigma_f$. Utilizing the wave functions of an ellipsoidal harmonic oscillator without spin-orbit coupling one can determine for which values of N, n_z, Ω, Λ, Σ the matrix elements of the electric and magnetic multipole moments differ from zero. The results of such a calculation are given in Tables I and II. The tables are arranged identically. The first column gives the difference $k = \Omega_f - \Omega_i$ between final and initial Ω. It

TABLE I. Selection rules for N, n_z, Λ, Σ in electric transitions of multipolarity λ

$\Delta\Omega = \Omega_f - \Omega_i$	$\Delta\Lambda = \Lambda_f - \Lambda_i$	$\Delta\Sigma = \Sigma_f - \Sigma_i$	$\Delta N = N_f - N_i$	$\Delta n_z = n_{zf} - n_{zi}$	Additional Conditions
$\pm\lambda$	$\pm\lambda$	0	$\lambda, \lambda-2, \ldots -\lambda$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda, \lambda-2, \ldots -\lambda$	±1	
0	0	0	±2	±2	$\lambda = 2$
0	0	0	$0, \pm2$	0	$\lambda = 2$

TABLE II. Selection rules for N, n_z, Λ, Σ in magnetic transitions of multipolarity λ

$\Delta\Omega = \Omega_f - \Omega_i$	$\Delta\Lambda = \Lambda_f - \Lambda_i$	$\Delta\Sigma = \Sigma_f - \Sigma_i$	$\Delta N = N_f - N_i$	$\Delta n_z = n_{zf} - n_{zi}$	Additional Conditions
$\pm\lambda$	$\pm\lambda$	0	$\lambda+1, \lambda-1, \ldots -\lambda-1$	±1	
$\pm\lambda$	$\pm(\lambda-1)$	±1	$\lambda-1, \lambda-3, \ldots -\lambda+1$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda-1, \lambda-3, \ldots -\lambda+1$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda+1, \lambda-1, \ldots -\lambda-1$	±2	$\lambda > 1$
$\pm(\lambda-1)$	$\pm(\lambda-2)$	±1	$\lambda-1, \lambda-3, \ldots -\lambda+1$	±1	$\lambda > 1$
0	0	0	$\pm1, \pm3$	±1	$\lambda = 2$
0	±1	∓1	±1	0	$\lambda = 2$

suffices to consder for a certain polarity λ transitions with $\Delta\Omega = \pm\lambda, \pm(\lambda - 1)$. An exception is quadrupole radiation in transitions with $\Omega_i = \Omega_f$. In the second and third columns the selection rules for Λ and Σ are given. They have to be considered together with $\Delta\Omega$. The condition $\Delta\Omega = \Delta\Lambda + \Delta\Sigma$ has to be fulfilled. We note that electric multipole transitions can occur only without change of the projection of the spin on the nuclear axis (neglecting the magnetic moment of the particle). Magnetic transitions can occur both with and without change of the projection of the spin on the nuclear axis. The fourth and fifth columns give the selection rules of N and n_z. If $\Delta n_z \neq 0$, the sign of Δn_z is restricted to the sign of ΔN for the largest $|\Delta N|$. The smaller $|\Delta N|$ can occur with either sign of Δn_z. For example, in an E1 transition with $\Delta\Omega = 0$, $\Delta\Lambda = 0$; the value $\Delta N = +1$ corresponds to $\Delta n_z = +1$, and for $\Delta N = -1$ only $\Delta n_z = -1$ is allowed. Or, in a magnetic transition with $\Delta\Omega = \pm(\lambda - 1)$, $\Delta\Lambda = \pm(\lambda - 1)$, we have $\Delta n_z = 2$ for $\Delta N = \lambda + 1$, and $\Delta n_z = -2$ for $\Delta N = -(\lambda + 1)$. For the other possible $\Delta N = \lambda - 1, \lambda - 3, \ldots, -\lambda + 1$, $\Delta n_z = \pm2$ is allowed.

It should be emphasized that the obtained selection rules hold only for transitions of the type $\Omega_f + \Omega_i > \lambda$. In transitions $\Omega_f + \Omega_i \leq \lambda$ the particular case with $\Omega = \frac{1}{2}$ appears. In this state the interaction with the surface vanishes and the spin-orbit interaction is not weakened. The quantum numbers n_z, Λ, Σ here have no meaning.

3. COMPARISON WITH EXPERIMENT

We shall now investigate how well the selection rules established in the previous section are obeyed in transitions in actual nuclei which have only finite deformations. For this it is necessary to compute transition probabilities using wave functions of deformed nuclei and to compare these with corresponding transition probabilities in spherical nuclei and with experiment. Such computations were performed for transitions in a number of rare-earth and transuranic elements for which there exist data on experimental half-lives.[7] The transition probabilities in deformed nuclei were calculated using Nilsson's[4] formulae and wave functions. The transition probabilities for spherical nuclei were computed with formulae given by Blatt and Weisskopf.[1] The results are given in Tables III and IV. Table III gives the results for electric multipole transitions and Table IV for magnetic multipole transitions. The first columns of the tables list the elements; the second the transition energy in kev, and the third the spin and parity of the initial (top) and final (bottom) state. Only states with $I = K = \Omega$ were considered. Transitions into rotational levels can easily be eliminated by use of the intensity rule given by Alaga et al.[6] The fourth column also gives two sets of data: the upper line has N, n_z, Λ, Σ for the initial state and the lower for the final state. In the fifth column the character of the transition is indicated. Transitions which are allowed with respect to all quantum numbers are uninhibited and denoted by u; transitions forbidden because of n_z, Λ, or Σ are hindered and denoted by h. In order to facilitate the determination of the character of the transition the table is broken up into parts according to the values of λ and k. The head of each subdivision lists the changes in the quantum numbers N, n_z, Λ, Σ which are allowed according to Table II. The sixth column lists the values of F_1 — the ratios of the theoretical transition probabilities in deformed nuclei to similar ones in spherical nuclei. And, finally, the last column shows the ratios of the experimental[7,8] to the theoretical transition probabilities in deformed nuclei, F_2. The ratio of the experimental transition probability to the transition probability in spherical nuclei (the F-factor) is equal to the product F_1F_2.

From Tables III and IV it is first evident that the quantum numbers n_z, Λ, and Σ play a significant part in real nuclei. If the transition is uninhibited the unified model and the shell model lead to similar results ($F \sim 1$). But if the transition is hindered then the unified model gives smaller transition probabilities than the shell model. The size of the decrease depends on the degree of forbiddenness. One sees from the given data that a first-order violation of the selection rules for one of the asymptotic quantum numbers decreases the transition probability by a factor 10 — 100. This allows an explanation of the disagreement between the experimental transition probabilities and the shell model (compare F_2 and F_1F_2). Unfortunately only little experimental data exists. However, even the presently-available data clearly indicate the usefulness of the unified model as can be seen from the last column of the tables. The unified model describes the experimental data much better than the shell model.

TABLE III. Electric multipole transitions ($E\lambda$)

Element	Transition energy kev	$\Omega_i\ \pi_i$ $\Omega_f\ \pi_f$	$N_i\ n_{zi}\Lambda_i\ \Sigma_i$ $N_f\ n_{zf}\Lambda_f\ \Sigma_f$	Transition character	$F_1 = \frac{T_{def}}{T_{sph}}$	$F_2 = \frac{T_{exp}}{T_{def}}$
$\lambda = 1$;		$\Delta\Omega = 0$;	$\Delta N = \pm 1$; $\Delta n_z = \pm 1$;		$\Delta\Lambda = 0$;	$\Delta\Sigma = 0$
Eu^{153}_{63}	97.3	$^5/_2-$ $^5/_2+$	$5\,3\,2+^1/_2$ $4\,1\,3-^1/_2$	*h*	$0{,}38 \cdot 10^{-4}$	
Np^{237}_{93}	60	$^5/_2-$ $^5/_2+$	$5\,2\,3-^1/_2$ $6\,4\,2+^1/_2$	*h*	$0.70 \cdot 10^{-4}$	0.13
Pu^{239}_{94}	106.2	$^5/_2-$ $^5/_2+$	$5\,0\,3-^1/_2$ $6\,2\,2+^1/_2$	*h*	$1.1 \cdot 10^{-6}$	0.5
$\lambda = 1$;		$\Delta\Omega = -1$; $\Delta N = \pm 1$;	$\Delta n_z = 0$,		$\Delta\Lambda = -1$;	$\Delta\Sigma = 0$
Lu^{177}_{71}	146	$^9/_2-$ $^7/_2+$	$5\,1\,4+^1/_2$ $4\,0\,4-^1/_2$	*h*	$0.40 \cdot 10^{-4}$	$0.7 \cdot 10^{-2}$
$\lambda = 2$;		$\Delta\Omega = 1$;	$\Delta N = 0$; $\Delta n_z = \pm 1$;		$\Delta\Lambda = 1$;	$\Delta\Sigma = 0$
Ta^{181}_{73}	482	$^5/_2+$ $^7/_2+$	$4\,0\,2+^1/_2$ $4\,0\,4-^1/_2$	*h*	$0.60 \cdot 10^{-4}$	
$\lambda = 2$;		$\Delta\Omega = 2$;	$\Delta N = 0$; $\Delta n_z = 0$;		$\Delta\Lambda = 2$;	$\Delta\Sigma = 0$
Ta^{181}_{73}	133	$^1/_2+$ $^5/_2+$	$4\,1\,1-^1/_2$ $4\,0\,2+^1/_2$	*h*	$1{,}8 \cdot 10^{-3}$	0,99
Eu^{153}_{63}	172	$^1/_2+$ $^5/_2+$	$4\,1\,1-^1/_2$ $4\,1\,3-^1/_2$	*u*	0,16	
$\lambda = 3$;		$\Delta\Omega = 3$;	$\Delta N = \pm 1$; $\Delta n_z = 0$;		$\Delta\Lambda = 3$;	$\Delta\Sigma = 0$
Dy^{165}_{66}	108	$^1/_2-$ $^7/_2+$	$5\,2\,1-^1/_2$ $6\,3\,3+^1/_2$	*h*	$0.85 \cdot 10^{-3}$	0,60*
Er^{167}_{68}	208	$^1/_2-$ $^7/_2+$	$5\,2\,1-^1/_2$ $6\,3\,3+^1/_2$	*h*	$0.80 \cdot 10^{-3}$	2.7*

* Computed by Iu. I. Kharitonov

The case Ir^{191}_{77} is of particular interest. This nucleus is on the boundary of the region of large deformations and the question of its equilibrium shape is presently open. An M1 transition of 82.6 kev was considered, with the deformation parameter taken as 0.14. The obtained result (Table IV) shows that the assumption of a deformed nucleus leads to a considerable improvement.

In conclusion we remark that the simple physical model of a single particle moving in a deformed harmonic oscillator potential on which this calculation is based gives a fundamentally correct description

TABLE IV. Magnetic multipole transitions (Mλ)

Element	Transition energy, kev	$\Omega_i\pi_i$ $\Omega_f\pi_f$	$N_i\ n_{zi}\ \Lambda_i\ \Sigma_i$ $N_f\ n_{zf}\ \Lambda_f\ \Sigma_f$	Transition character	$F_1 = \frac{\tau_{\text{def}}}{\tau_{\text{sph}}}$	$F_2 - \frac{\tau_{\text{exp}}}{\tau_{\text{def}}}$
$\lambda = 1;$	$\Delta\Omega = 1;$	$\Delta N = 0;$	$\Delta n_z = \pm\begin{matrix}0,\\1;\end{matrix}$		$\Delta\Lambda = \begin{matrix}0,\\1;\end{matrix}$	$\Delta\Sigma = \begin{matrix}1\\0\end{matrix}$
Eu_{63}^{153}	69	$^1/_2+$ $^3/_2+$	$4\,1\,1-{}^1/_2$ $4\,1\,1+{}^1/_2$	u	0.72	0.09
Eu_{63}^{153}	103	$^3/_2+$ $^5/_2+$	$4\,1\,1+{}^1/_2$ $4\,1\,3-{}^1/_2$	h	$1{,}2\cdot10^{-4}$	13
Ta_{73}^{181}	482	$^5/_2+$ $^7/_2+$	$4\,0\,2+{}^1/_2$ $4\,0\,4-{}^1/_2$	h	$1{,}4\cdot10^{-3}$	$0{,}87\cdot10^{-2}$
Ir_{77}^{191}	82.6	$^1/_2+$ $^3/_2+$	$4\,0\,0+{}^1/_2$ $4\,0\,2-{}^1/_2$	h	$0{,}56\cdot10^{-3}$	1,8
$\lambda = 2$	$\Delta\Omega = -1;$	$\Delta N = -1;$	$\Delta n_z = -1;$		$\Delta\Lambda = 0;$	$\Delta\Sigma = -1$
Lu_{71}^{177}	146	$^9/_2-$ $^7/_2+$	$514+{}^1/_2$ $404-{}^1/_2$	u	0,1	

of the laws of the electromagnetic transitions in deformed nuclei. This is a rather remarkable fact.

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SCATTERING OF SLOW NEUTRONS IN FERROMAGNETS

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The scattering of slow neutrons in ferromagnets is analyzed, using the theory of spin-waves. The analysis is applied to single crystals and to polycrystals.

1. INTRODUCTION

THIS paper will deal with the scattering of slow neutrons in ferromagnets, at low temperatures where the behavior of the ferromagnet is described by spin-wave theory. This problem has been recently discussed by several authors.[1-4] The most complete discussion is that by Elliott and Lowde. They consider mainly the inelastic scattering of neutrons in a single crystal, the scattering being accompanied by the emission or absorption of a spin-wave of small momentum, and the temperature being not too low so that the energy-spectrum of spin-waves is given by the Rayleigh-Jeans formula. The case of greatest physical interest, the diffuse scattering close to a Bragg reflection, is described satisfactorily by their theory. However, the assumptions which they make are not necessary; the cross-sections can be calculated without any such assumptions, and the results remain simple in form.

We find that, when the neutron wave-length is much greater than the lattice-constant, the large-angle scattering of a neutron is accompanied by the absorption of a spin-wave of large momentum for which the Rayleigh-Jeans distribution is usually incorrect. Therefore the statement[4,5] that the long-wave-length inelastic scattering cross-section is proportional to temperature becomes correct only at temperatures low compared with the ferromagnetic exchange interaction.

In this paper we calculate the elastic scattering, and the inelastic scattering in single crystals and polycrystals with absorption or emission of one-spin wave, and we also estimate the magnitude of processes in which more than one spin-wave is emitted or absorbed.

2. DERIVATION OF THE CROSS-SECTION FORMULAE

We consider neutrons in a ferromagnet being scattered by magnetic interaction with the atomic spins. We do not include the scattering due to the interaction with the orbital part of the atomic magnetic moments, since this scattering is identical with the scattering in a paramagnetic medium and was calculated by Migdal.[5] Because the orbital moments are randomly oriented, there is no interference between the orbital and spin contributions to the scattering. For a detailed investigation of inelastic scattering we need to know the wave-functions of the scatterer; we are therefore restricted to the low range of temperatures (far below the Curie temperature) in which a detailed theory of ferromagnetism, the spin-wave theory, is valid.

The matrix elements of the interaction of a neutron with the spins of a system of N identical atoms have been calculated by Halpern and Johnson.[6] They have the form

$$V_{\mathbf{p}na}^{\mathbf{p}'n'a'} = -\frac{4\pi\hbar^2}{M} r_0 \gamma F(q) \left(n'a' \left| \sum_l e^{i\mathbf{q}\mathbf{R}_l} (\mathbf{S}_l, \mathbf{S} - (\mathbf{e}\mathbf{S})\mathbf{e}) \right| na\right), \tag{1}$$

Here $\mathbf{p}$ is the neutron wave-vector, n is one component of its spin, and a is the state of the ferromagnet, initially. In the final state the corresponding quantities are $\mathbf{p}'$, n', a'. γ is the neutron magnetic moment in nuclear magnetons, $\mathbf{S}$ is its spin, $r_0 = e^2/mc^2$, M is the neutron mass, $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, $\mathbf{e} = \mathbf{q}/q$, $\mathbf{S}_\ell$ and $\mathbf{R}_\ell$ are the spin and position of atom number ℓ,

$$F(q) = \int d\tau \Psi_l^* \sum_{\nu=1}^{z} \frac{e^{i\mathbf{q}\rho_\nu} \mathbf{s}_\nu \mathbf{S}_l}{S_l(S_l+1)} \Psi_l, \tag{2}$$

Ψ_ℓ is the wave-function of atom number ℓ, $\mathbf{s}_\nu$ and $\boldsymbol{\rho}_\nu$ are the spin and position of electron number ν, and the sum extends over the z electrons in the atom.

The matrix elements of the operator $\sum \exp(i\mathbf{q}\mathbf{R}_\ell)\mathbf{S}_\ell$ can be easily calculated from spin-wave theory in the form introduced by Dyson.[7,8] The operators S_ℓ^z and $S_\ell^\pm = S_\ell^x \pm iS_\ell^y$ are connected with spin-wave absorption and creation operators in the following way (see the Appendix).

$$S_l^z = -S\sum_\mu e^{-i\mu\mathbf{R}_l}\Big[\delta_{\mu 0} - (SN)^{-1}\sum_\lambda \alpha_{\mu+\lambda}^+\alpha_\lambda\Big], \tag{3a}$$

$$S_l^+ = (2SN^{-1})^{1/2}\sum_\mu e^{-i\mu\mathbf{R}_l}\alpha_\mu^+, \tag{3b}$$

$$S_l^- = (2SN^{-1})^{1/2}\sum_\mu e^{-i\mu\mathbf{R}_l}\Big[\alpha_{-\mu} - (2SN)^{-1}\sum_{\lambda,\sigma}\alpha_{\lambda+\sigma+\mu}^+\alpha_\lambda\alpha_\sigma\Big], \tag{3c}$$

where the z-axis is along the direction of magnetization of a domain, α_μ and α_μ^+ are absorption and emission operators for a spin-wave with wave-vector $\boldsymbol{\mu}$ and energy

$$\varepsilon_\mu = JS\sum_\delta(1 - e^{i\mu\delta}) \approx \frac{1}{6}JS\gamma_0\delta^2\mu^2, \tag{4}$$

J is the ferromagnetic exchange integral, $\boldsymbol{\delta}$ is a vector joining any atom of the lattice to one of its nearest neighbors, and γ_0 is the number of nearest neighbors over which the δ-summation extends.*

Equation (3) implies the existence of the following types of scattering. (1) Elastic. (2) With emission of one spin-wave. (3) With absorption of one spin-wave. (4) With emission of one and absorption of one spin-wave. (5) With emission of one and absorption of two spin-waves. The corresponding cross-sections are

1)
$$\frac{d\sigma_0}{d\Omega} = S^2r_0^2\gamma^2F^2(q)\frac{1}{N}\Big|\sum_l e^{i\mathbf{q}\mathbf{R}_l}\Big|^2\Big\langle\Big(1 - \frac{1}{NS}\sum_\lambda a_\lambda\Big)^2\Big\rangle e^{-2W_\mathbf{q}}(1 - e_z^2); \tag{5a}$$

2)
$$\frac{d\sigma_{+1}}{d\Omega} = \frac{1}{2}Sr_0^2\gamma^2F^2(q)\frac{1}{N}\Big|\sum_l e^{i(\mathbf{q}-\mu)\mathbf{R}_l}\Big|^2\frac{v_0}{(2\pi)^3}e^{-2W_\mathbf{q}}\frac{p'}{p}(1 + e_z^2)\,]\langle a_\mu\rangle + 1]\,d\boldsymbol{\mu}; \tag{5b}$$

3)
$$\frac{d\sigma_{-1}}{d\Omega} = \frac{1}{2}Sr_0^2\gamma^2F^2(q)\frac{1}{N}\Big|\sum_l e^{i(\mathbf{q}+\mu)\mathbf{R}l}\Big|^2\frac{v_0}{(2\pi)^3}e^{-2W_\mathbf{q}}\frac{p'}{p}(1 + e_z^2)\Big\langle a_\mu\Big(1 - \sum_{\lambda\neq\mu}a_\lambda/NS\Big)\Big\rangle d\mu; \tag{5c}$$

4)
$$\frac{d\sigma_{+1,-1}}{d\Omega} = r_0^2\gamma^2F^2(q)\frac{1}{N}\Big|\sum_l e^{i(\mathbf{q}+\mu-\nu)\mathbf{R}_l}\Big|^2\frac{v_0^2}{(2\pi)^6}e^{-2W_\mathbf{q}}\frac{p'}{p}(1 - e_z^2)\langle a_\mu(a_\nu + 1)\rangle\,d\boldsymbol{\mu}\,d\boldsymbol{\nu}; \tag{5d}$$

5)
$$\frac{d\sigma_{+1,-2}}{d\Omega} = \frac{1}{2S}r_0^2\gamma^2F^2(q)\frac{1}{N}\Big|\sum_l e^{i(\mathbf{q}+\nu+\sigma-\mu)\mathbf{R}_l}\Big|^2 e^{-2W_\mathbf{q}}\frac{v_0^3}{(2\pi)^9}\frac{p'}{p}\langle a_\nu a_\sigma(a_\mu + 1)\rangle(1 + e_z^2)\,d\boldsymbol{\mu}\,d\boldsymbol{\nu}\,d\boldsymbol{\sigma}. \tag{5e}$$

Here v_0 is the volume of the unit cell, $e^{-2W_\mathbf{q}}$ is the usual temperature factor, a_μ is the number of spin-waves with wave-vector $\boldsymbol{\mu}$, and $< >$ denotes an average over the initial states of the scatterer, so that $< a_\mu >$ is the Planck distribution-function which we shall denote by $n(\mu)$.

In this paper we study the cross-sections (5a) – (5c). A detailed investigation of (5d) and (5e) will be made later.

3. ELASTIC SCATTERING

Consider Eq. (5a). At low temperatures $\Sigma a_\mu/NS \ll 1$, and we may neglect the square of this quantity. Its mean value is just equal to the deviation of the spontaneous magnetization at temperature T from its value at T = 0. When the number N of atoms in the crystal is large,

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*We consider only ferromagnets of cubic symmetry.

$$\frac{1}{N}\left|\sum_{l}\exp(i\mathbf{q}\mathbf{R}_l)\right|^2=\frac{(2\pi)^3}{v_0}\sum_{\tau}\delta(\mathbf{q}-\boldsymbol{\tau}), \tag{6}$$

where τ is a vector of the reciprocal lattice multiplied by 2π. Therefore Eq. (5a) becomes

$$\frac{d\sigma_0}{d\Omega}=S^2r_0^2\gamma^2F^2(q)\,e^{-2W_q}\,[1-2G(T)]\,(1-e_z^2)\,\frac{(2\pi)^3}{v_0}\sum_{\tau}\delta(\mathbf{q}-\boldsymbol{\tau}). \tag{7}$$

This expression was obtained by Van Hove[9] by a different method. From spin-wave theory we have

$$G(T)=\zeta({}^3/_2)\,(T/2\pi T_c)^{3/2}+O\,[(T/2\pi T_c)^{5/2}], \tag{8}$$

where $\zeta(x)$ is the Riemann zeta-function, $\nu=\delta^2 v_0^{-2/3}$, and $T_C=JS\gamma_0\nu/3k$ is approximately equal to the Curie temperature.[8] Equation (8), like the spin-wave theory and like our whole treatment of neutron scattering, is valid, as shown for example by Dyson,[8] when $T \ll T_C/2\nu$.

A detailed theory of ferromagnetism is not necessary to derive Eq. (7). In fact Eq. (7) follows directly from Eq. (1), if we use the following three pieces of information. (a) The translational symmetry of the crystal implies that the matrix element $(a\,|\,S_\ell^z\,|\,a)$ is independent of ℓ. (b) The definition of $G(t)$ requires

$$S^{-2}\,\langle(a\,|\,S_l^z\,|\,a)^2\rangle\approx 1-2G(T)$$

(c) The matrix elements $(a\,|\,S_\ell^+\,|\,a)$ are zero, since the operator S_ℓ^+ changes by one unit the z-component of the total spin and this z-component is a constant of the motion. Therefore Eq. (7) can easily be generalized to the case of scattering by antiferromagnets. In the simplest case, an antiferromagnetic crystal can be considered as composed of two equivalent sub-lattices, interpenetrating each other and magnetized in opposite directions. The only difference from the ferromagnetic case is that the matrix elements $(a\,|\,S_\ell^z\,|\,a)$ reverse their signs when we go from one atom to any of its nearest neighbors, the absolute values still being equal. Accordingly we may write

$$(a\,|\,S_l^z\,|\,a)=(a\,|\,S_l^z\,|\,a)'\exp(i\mathbf{w}\mathbf{R}_l),$$

where $(a\,|\,S_\ell^z\,|\,a)'$ is independent of ℓ, and $\mathbf{w}$ is any vector of the reciprocal sub-lattice which does not belong to the reciprocal lattice. Thus Eq. (7) remains valid for the scattering by an antiferromagnet, if we replace the vectors τ by $\mathbf{w}$, and define $G(t)$ with reference to either sub-lattice. By observing the scattering of neutrons in antiferromagnets we may obtain information about the temperature-dependence of the spontaneous magnetization in the sublattices.

Averaging Eq. (7) over the direction of τ, we obtain the scattering cross-section in a polycrystal

$$\frac{d\sigma_0}{d\Omega}=S^2r_0^2\gamma^2\,[1-2G(T)]\,\frac{(2\pi)^3}{v_0}\sum_{\tau}\frac{g_\tau F^2(\tau)}{4\pi\tau^2}\,(1-e_z^2)\,e^{-2W_\tau}\,\delta(q-\tau), \tag{9}$$

where g_τ is the number of distinct reciprocal lattice-vectors with length τ, and

$$e_z^2=\left[\cos\frac{\theta_0}{2}\sin\varphi\sin\zeta-\sin\frac{\theta_0}{2}\cos\zeta\right]^2, \tag{10}$$

where θ_0 is the scattering angle which satisfies the condition $\sin(\theta_0/2)=\tau/2p$, ζ is the angle between $\mathbf{p}$ and the z-axis, and φ is the polar angle of the vector $\mathbf{p}'$ measured in the plane perpendicular to $\mathbf{p}$ from the direction $[\mathbf{p}\times\mathbf{z}]$.

The total elastic scattering cross-section in a polycrystal is

$$\sigma_0=S^2r_0^2\gamma^2\,[1-2G(T)]\,\frac{(2\pi)^3}{v_0p^2}\sum_{\tau\leqslant 2p}\frac{g_\tau F^2(\tau)}{4\tau}\,e^{-2W_\tau}\left[1+\cos^2\zeta+\frac{\tau^2}{4p^2}(1-3\cos^2\zeta)\right]. \tag{11}$$

4. SINGLE-SPIN-WAVE SCATTERING IN A SINGLE CRYSTAL

Starting from Eqs. (5b), (5c), and (6), and neglecting $\Sigma\, a_\mu/NS$ in comparison with unity in Eq. (5c), we obtain the cross-section for single-spin-wave scattering in a single crystal,[*]

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[*] Equation (12) coincides with Elliott and Lowde's formula[4] for the single-spin-wave scattering, if we replace $n(\mu)+{}^1/_2\pm{}^1/_2$ by $6kT/JS\gamma_0\delta^2\mu^2$.

$$d\sigma^{\tau}_{\pm 1}/d\Omega = \frac{1}{2} S r_0^2 \gamma^2 F^2(q)\, e^{-2W_q}(1+e_z^2)\frac{p'}{p}[n(\mu)+{}^1/_2 \pm {}^1/_2]\,\delta(\mathbf{q}\pm\boldsymbol{\mu}+\boldsymbol{\tau})\,d\boldsymbol{\mu}. \tag{12}$$

The vectors $\mathbf{p}'$, $\mathbf{p}$, and $\boldsymbol{\mu}$ are connected by the energy conservation law:

$$p'^2 = p^2 \mp (2M/\hbar^2)\,\varepsilon(\mu) \approx p^2 \mp \alpha\mu^2. \tag{13}$$

Here the minus sign holds for scattering with emission of a spin-wave, the plus sign for scattering with absorption. For ion $\alpha \sim 100$, and in general $\alpha \gg 1$. In Eq. (12) we may replace $\mathbf{q}$ by $\boldsymbol{\tau}$ everywhere except in the argument of the δ-function. This is because the whole spin-wave theory, and Eq. (13) in particular, only holds for $\mu\delta < 1$.

We study first the case of absorptive scattering. Integrating Eq. (12) with respect to μ and using Eq. (13), we find

$$d\sigma^{\tau}_{-1}/d\Omega = (d\sigma^{\tau}_{-1}/d\Omega)_+ + (d\sigma^{\tau}_{-1}/d\Omega)_-;\quad (d\sigma^{\tau}_{-1}/d\Omega)_{\pm} = \frac{1}{2} S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau}(1+(\tau\mathbf{m})^2/\tau^2)\, n(\mu_{\pm}) \tag{14}$$

$$\times \frac{P\alpha}{p(\alpha-1)^2}\,\frac{[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \cos^2\vartheta_0}]^2}{\sqrt{\cos^2\vartheta-\cos^2\vartheta_0}}; \tag{15}$$

$$\cos^2\vartheta_0 = (\alpha-1)\,\alpha^{-1}(1+p^2/\alpha P^2); \tag{16}$$

$$\mu^2_{\pm}(\cos\vartheta) = \frac{P^2\alpha}{(\alpha-1)^2}\left[\left(\cos\vartheta \pm \sqrt{\cos^2\vartheta-\cos^2\vartheta_0}\right)^2 - \left(\frac{\alpha-1}{\alpha}\right)^2\frac{p^2}{P^2}\right], \tag{17}$$

Here ϑ is the scattering angle measured from the direction $\mathbf{P} = \mathbf{p} + \boldsymbol{\tau}$, and $\mathbf{m}$ is a unit vector in the direction of magnetization. The magnitude of the momentum of the scattered neutron is given by

$$p'_{\pm} = \frac{\alpha P}{\alpha-1}(\cos\vartheta \pm \sqrt{\cos^2\vartheta - \cos^2\vartheta_0}), \tag{18}$$

and the range of variation of $\cos\vartheta$ is

$$1 \geqslant \cos\vartheta \geqslant \cos\vartheta_0 > 0. \tag{19}$$

Since $\cos^2\vartheta_0 < 1$, it follows that

$$\frac{p^2}{P^2} < \frac{\alpha}{\alpha-1} \text{ for } \cos\Psi > -\frac{1}{2}\left(\frac{\tau}{p} + \frac{1}{\alpha}\frac{p}{\tau}\right), \tag{20}$$

where Ψ is the angle between $\mathbf{p}$ and $\boldsymbol{\tau}$. Equation (16) implies $\cos^2\vartheta_0 > 1 - (1/\alpha)$, and since $\alpha \gg 1$, also $\vartheta < \alpha^{-1/2}$. Thus the scattering with absorption of a spin-wave occurs only in a narrow cone around the axis $\mathbf{P} = \mathbf{p} + \boldsymbol{\tau}$. To each direction of scattering within this cone there correspond two values given by Eq. (18) for the scattered neutron momentum.

By Eq. (17), $\mu^2_+(\cos\vartheta)$ decreases and $\mu^2_-(\cos\vartheta)$ increases with increasing ϑ. Thus

$$\mu^2_+(1) > \mu^2_+(\cos\vartheta_0) = \mu^2_-(\cos\vartheta_0) > \mu^2_-(1). \tag{21}$$

From Eqs. (14), (15), and (21) we deduce the formula for the total cross-section

$$\sigma^{\tau}_{-1} = \frac{1}{2}\pi S r_0^2\gamma^2 F^2(\tau)\left[1+\frac{(\tau\mathbf{m})^2}{\tau^2}\right] e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{1-\exp\{-T_c\delta^2\mu^2_+(1)(2\nu T)^{-1}\}}{1-\exp\{-T_c\delta^2\mu^2_-(1)(2\nu T)^{-1}\}}. \tag{22}$$

We next study a few limiting cases.

I. $\mathbf{P} = \mathbf{p}$. Then Eqs. (16) and (17) give

$$\cos^2\vartheta_0 = 1-\alpha^{-2},\quad \mu^2_+(1) = 4p^2/(\alpha-1)^2,\quad \mu^2_-(1) = 0.$$

If also p and T satisfy

$$4p^2\delta^2/(\alpha-1)^2 \ll 2\nu T/T_c,$$

then $n(\mu_+)$ can be approximated by $< 2\nu T/T_c > \mu^2_+\delta^2$ in Eq. (15), and the cross-section becomes proportional to temperature. As $\vartheta \to 0$ the cross-section tends to infinity. However, Elliott and Lowde[4] have shown that the infinity does not actually occur, since the minimum wave-vector of a spin-wave is different from zero and is of the order of magnitude $2\pi/\delta N^{1/3}$, where N is roughly the number of atoms in a domain. Therefore the cross-section becomes

$$\sigma^{\tau}_{-1} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathrm{m}\tau)^2/\tau^2]\, e^{-2W_\tau}\, \frac{2\nu T}{T_c}(p\delta)^{-2} \ln \frac{p^2\delta^2 N^{2/3}}{\pi^2(\alpha-1)^2}. \tag{23}$$

When $P = p$ there is also elastic scattering.

II. If $p \neq P$ but $\mu_+^2(1)\delta^2 \ll 2\nu T/T_c$, the cross-section remains proportional to temperature,

$$\sigma^{\tau}_{-1} = \frac{\pi}{2} S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathrm{m}\tau)^2/\tau^2]\, e^{-2W_\tau}\,(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\mu_+^2(1)}{\mu_-^2(1)}. \tag{24}$$

III. If $\mu_-^2(1)\delta^2 > 2\nu T/T_c$, the cross-section is proportional to

$$\exp\{-\mu_-^2(1)\,\delta^2 T_c/2\nu T\} - \exp\{-\mu_+^2(1)\,\delta^2 T_c/2\nu T\}.$$

But this expression is very small compared with unity, and so there is practically no neutron scattering with spin-wave absorption in this case.

The cross-section is a maximum when $p^2 = P^2$. As p^2 decreases, $\mu_-^2(1)$ increases and hence the cross-section decreases.

We next consider the scattering with spin-wave emission. As in the case of absorption, we find

$$d\sigma^{\tau}_{+1}/d\Omega = (d\sigma^{\tau}_{+1}/d\Omega)_+ + (d\sigma^{\tau}_{+1}/d\Omega)_-; \tag{25}$$

$$(d\sigma^{\tau}_{+1}/d\Omega)_\pm = \frac{1}{2} S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau}\,[1 + (\mathrm{m}\tau)^2/\tau^2]$$

$$\times\,[n(\mu'_\pm) + 1]\frac{P\alpha}{(\alpha+1)^2}\left[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)}\right]^2\left[\cos\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)\right]^{-1/2}; \tag{26}$$

$$\mu'^2_\pm(\cos\vartheta) = \frac{\alpha P^2}{(\alpha+1)^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)^2\frac{p^2}{P^2} - \left[\sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)} \pm \cos\vartheta\right]^2\right\}; \tag{27}$$

$$p'_\pm = \frac{\alpha P}{\alpha+1}\left[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)}\right]. \tag{28}$$

Scattering is now possible only when

$$\frac{p^2}{P^2} > \frac{\alpha}{\alpha+1} \quad \text{or} \quad \cos\Psi < -\frac{1}{2}\left(\frac{\tau}{p} - \frac{1}{\alpha}\frac{p}{\tau}\right). \tag{29}$$

Since $\cos\Psi \geq -1$, Eq. (29) implies that scattering will occur with spin-wave emission only when

$$p > \alpha\tau\left(\sqrt{1 + (1/\alpha)} - 1\right) \approx (\tau/2)\left(1 - (1/4\alpha)\right). \tag{30}$$

We need to consider two cases

$$1)\ \frac{\alpha+1}{\alpha}\left(1 - \frac{p^2}{\alpha P^2}\right) \equiv \cos^2\vartheta_1 > 0, \quad 2)\ 1 - \frac{p^2}{\alpha P^2} < 0.$$

We discuss case (1) first. As in the discussion of absorptive scattering, $\cos\vartheta$ varies within the range

$$1 \geqslant \cos\vartheta \geqslant \cos\vartheta_1 > 0 \tag{31}$$

and to every scattering angle ϑ correspond two outgoing neutron momenta given by Eq. (28). Also $\mu'^2_+(\cos\vartheta)$ increases and $\mu'^2_-(\cos\vartheta)$ decreases with increasing ϑ, so that

$$\mu'^2_-(1) > \mu'^2_-(\cos\vartheta_1) = \mu'^2_+(\cos\vartheta_1) > \mu'^2_+(1). \tag{32}$$

The total cross-section is given by

$$\sigma^{\tau}_{+1} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathrm{m}\tau)^2/\tau^2]\, e^{-2W_\tau}\,(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\exp\{T_c\,\delta^2\mu'^2_-(1)\,(2\nu T)^{-1}\} - 1}{\exp\{T_c\,\delta^2\mu'^2_+(1)\,(2\nu T)^{-1}\} - 1}. \tag{33}$$

As in the discussion of absorptive scattering, we consider three limiting sub-cases of case (1).

I. $P = p$. $\cos^2\vartheta_1 = 1 - \alpha^{-2}$, $\mu'^2_-(1) = 4p^2/(\alpha+1)^2$, $\mu'^2_+(1) = 0$, and, if we assume $4p^2\delta^2/(\alpha+1)^2 \ll 2\nu T/T_c$, then

$$\sigma^{\tau}_{+1} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathrm{m}\tau)^2/\tau^2]\, e^{-2W_\tau}\,(p\delta)^{-2}\frac{2\nu T}{T_c}\ln\frac{p^2\delta^2 N^{2/3}}{\pi^2(\alpha+1)^2} \tag{34}$$

So in this case $\sigma^t_{+1} \approx \sigma^t_{-1}$. This means that around the Bragg maximum of width $N^{-1/3}$ there is a diffuse maximum of width $1/\alpha$ caused by single-spin-wave inelastic scattering.

II. $p \neq P$ but $\mu'^2_-(1)\delta^2 \ll 2\nu T/T_c$. Then

$$\sigma^\tau_{+1} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (m\tau)^2/\tau^2]\,e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\mu'^2_-(1)}{\mu'^2_+(1)} \tag{35}$$

III. $\mu'^2_+(1)\delta^2 \gtrsim 2\nu T/T_c$. In this case the cross-section is given, with an error of order $\exp\{-T_c[\mu'_+(1)\delta]^2/2\nu T\}$, by the formula

$$\sigma^\tau_{+1} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (m\tau)^2/\tau^2]\,e^{-2W_\tau}\frac{4\alpha\sin\vartheta_1}{(\alpha+1)^2}, \tag{36}$$

and is very small when α is large. It is larger, the larger the ratio p^2/P^2. As in the case of absorptive scattering, the maximum cross-section is at $p^2/P^2 = 1$.

Now we return to case (2). This case is possible if

$$\tau\sqrt{\alpha}(\sqrt{\alpha}+1)^{-1} < p < \tau\sqrt{\alpha}(\sqrt{\alpha}-1)^{-1},$$

which means that p is very close to τ. Equation (28) then implies that $p'_- < 0$. There is thus only one outgoing neutron momentum p'_+ at each scattering angle, and the angle can very from zero to π. Also

$$\mu'^2_+(1) \leqslant \mu'^2_+(\cos\vartheta) \leqslant \mu'^2_+(-1). \tag{37}$$

Equation (27) implies that $\mu'^2_+(1) > p^2/\alpha \approx \tau^2/\alpha$, and so the cross-section is small at sufficiently low temperatures. When $\mu'^2_+(1)\delta^2 < 1$ the cross-section is given approximately by Eq. (36) with $\sin\vartheta_1$ replaced by $\sqrt{\frac{1}{\alpha}\left(\frac{\alpha+1}{\alpha}\frac{p^2}{P^2}\right) - 1}$.

From Eqs. (20) and (29) we see that scattering with emission of a spin-wave and scattering with absorption can occur simultaneously only when

$$-\left(\frac{\tau}{p} + \frac{1}{\alpha}\frac{p}{\tau}\right) \leqslant 2\cos\Psi \leqslant -\left(\frac{\tau}{p} - \frac{1}{\alpha}\frac{p}{\tau}\right), \tag{38}$$

which just defines the range of values of Ψ for which both cross-sections attain their maxima.

5. SINGLE-SPIN-WAVE SCATTERING IN POLYCRYSTALS

When Eq. (12) is averaged over the possible orientations of q and τ and integrated over the direction of μ, we obtain the total cross-section for scattering in a polycrystal with absorption or emission of one spin-wave with wave-vector μ,

$$d\sigma^\tau_{\pm 1}(p,\mu)/d\Omega = \frac{1}{4} S r_0^2 \gamma^2 F^2(\tau)\,e^{-2W_\tau}(1+e_z^2)\,\frac{g_\tau}{\tau}\frac{p'}{p}\left[n(\mu) + \frac{1}{2} \pm \frac{1}{2}\right]\frac{\mu\,d\mu}{q}, \tag{39}$$

with

$$\tau - \mu < q < \tau + \mu. \tag{40}$$

Since $q = |p' - p|$, $|p - p'| < q < p + p'$, and therefore the inequality (40) need be stated only when

$$|p - p'| < \tau - \mu, \tag{41a}$$

$$\tau + \mu < p + p'. \tag{41b}$$

We must in any case have

$$\tau + \mu > |p - p'|, \tag{41c}$$

$$\tau - \mu < p + p'. \tag{41d}$$

The inequalities (41c) and (41d) fix the limits within which q may vary when μ is given.

Consider first the absorptive scattering. We have to satisfy the inequalities (41a) – (41d) with p' given by Eq. (13). The allowed values of q are plotted in Fig. 1, the boundary cruves being given by

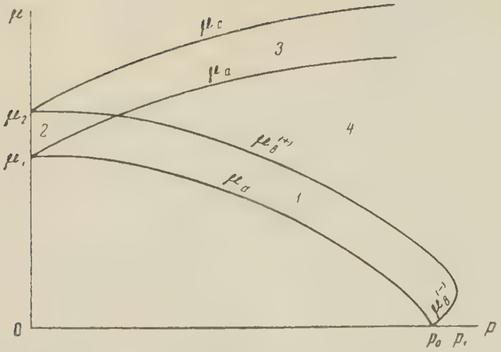

FIG. 1. Ranges of variation of q for given p and μ, for absorptive scattering. $p_0 = \tau/2$, $p_1 = \tau\alpha(1 - \sqrt{1-(1/\alpha)})$, $\mu_1 = \tau/(\sqrt{\alpha}+1)$, $\mu_2 = \tau/(\sqrt{\alpha}-1)$. In region (1) $\tau - \mu \le q \le p + p'$. In (2) $p' - p \le q \le p' + p$. In (3) $p' - p \le q \le \tau + \mu$. In (4) $\tau - \mu \le q \le \tau + \mu$. Outside these regions there is no scattering.

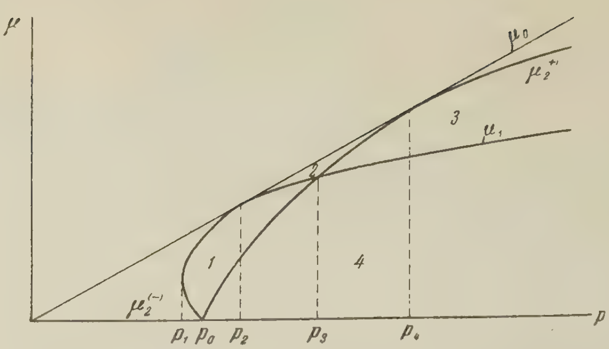

FIG. 2. Ranges of variation of q for given p and μ, for scattering with spin-wave emission. $p_1 = \tau\alpha(\sqrt{1+(1/\alpha)} - 1)$, $p_0 = \tau/2$, $p_2 = \sqrt{\alpha}\tau/(\sqrt{\alpha}+1)$, $p_3 = \tau$, $p_4 = \sqrt{\alpha}\tau/(\sqrt{\alpha}-1)$, $\mu_0 = p/\sqrt{\alpha}$. In region (1) $\tau - \mu \le q \le p + p'$. In (2) $p - p' \le q \le p' + p$. In (3) $p - p' \le q \le \tau + \mu$. In (4) $\tau - \mu \le q \le \tau + \mu$. Outside these regions there is no scattering.

$$\mu_a = \frac{-\tau - p + \sqrt{\alpha\tau^2 + 2\alpha\tau p + p^2}}{\alpha - 1}; \quad \mu_c = \frac{\tau + p + \sqrt{\alpha\tau^2 + 2\alpha\tau p + p^2}}{\alpha - 1};$$
$$\mu_b^{(\pm)} = \frac{\tau - p \pm \sqrt{\alpha\tau^2 - 2\alpha\tau p + p^2}}{\alpha - 1}; \quad \mu_d = \frac{p - \tau + \sqrt{\alpha\tau^2 - 2\alpha\tau p + p^2}}{\alpha - 1}. \tag{42}$$

The range of variation of q is of no interest when μ is large, since $n(\mu)$ is then small. Thus the allowed ranges of q when p, μ, τ are given can for practical purposes be taken as

$$1)\ \tau - \mu \leqslant q \leqslant \tau + \mu; \quad 2)\ \tau - \mu \leqslant q \leqslant p + p'.$$

The differential cross-section is

$$d\sigma_{-1}^{\tau} / d\mu = (4\pi/3)\, S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau / \tau p^2)\, n(\mu)\, \mu^2 \tag{43a}$$

in case (1) and

$$d\sigma_{-1}^{\tau} / d\mu = (2\pi/3)\, S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau / \tau p^2)\, n(\mu)\, \mu\, (p + p' - \tau + \mu) \tag{43b}$$

in case (2). These results are calculated for an unmagnetized crystal, setting e_z^2 equal to $\frac{1}{3}$. If $p \ge \tau\alpha(1 - \sqrt{(1-(1/\alpha)})$, only case (1) can occur, and the total cross-section becomes

$$\sigma_{-1}^{\tau}(p) = \frac{1}{3} S^2 r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} \frac{g_\tau (2\pi)^3}{\tau p^2 v_0} G(T). \tag{44}$$

When $\tau/2 < p < \alpha\tau(1 - \sqrt{1-(1/\alpha)})$, we can compute the correction to Eq. (44) produced by the range of μ for which $d\sigma_{-1}^{\tau}/d\mu$ is given by Eq. (43b) rather than by Eq. (43a). However, this correction is of order $1/\alpha$ and therefore small when α is large. Equation (44) can be taken as correct all the way down to $p = \tau/2$. Further, using the relation $p + p' + \mu - \tau < 2\mu$, we can show that when $\mu_d < \mu < \mu_b^{(+)}$, Eq. (44) remains valid for all p satisfying $\mu_d \delta \ll \sqrt{2\nu T/T_c}$. When p decreases still further the cross-section begins to fall off rapidly.

Next we discuss the scattering with spin-wave emission. When $\mu\delta < 1$, so that the relation between energy and momentum of a spin-wave is quadratic, the values of q allowed by the inequalities (41) are plotted in Fig. 2. The boundary curves are

$$\mu_1^{(\pm)} = \frac{\tau - p \pm \sqrt{p^2 + 2\alpha\tau p - \alpha\tau^2}}{\alpha + 1}; \quad \mu_2 = \frac{p - \tau + \sqrt{p^2 + 2\alpha\tau p - \alpha\tau^2}}{\alpha + 1}.$$

The cross-section is then

$$d\sigma_{+1}^{\tau} / d\mu = (2\pi S/3)\, r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau / \tau p^2)\, [n(\mu) + 1]\, \mu \Delta q, \tag{45}$$

where Δq is the difference between the maximum and minimum values of q which are allowed for given p, μ. Again the scatterer is assumed unmagnetized.

We now consider the total cross-section for scattering with emission of one spin-wave. When p is close to $\tau/2$, the total cross-section is obviously small. It increases rapidly with p. When $p \sim \tau$, the part of the total cross-section with spin-wave emission which depends on $n(\mu)$ becomes equal to the absorptive cross-section given by Eq. (44). As p increases further, the analytic form of the cross-section remains unchanged. The part of the cross-section independent of $n(\mu)$ can be easily evaluated in the case when the maximum value of μ for given p does not exceed $1/\delta$. When $p \sim \tau$, this part becomes

$$(2\pi/3)\, S r_0^2\, \gamma^2 F^2(\tau)\, e^{-2W_\tau}\, (g_\tau/\tau p^2)\, (\tau/\sqrt{\alpha})^3. \tag{46}$$

As p increases further, the condition $\delta\mu < 1$ ceases to hold. But for sufficiently large p, (when $p\delta > [(\tau\delta)^2 + \alpha]/2\tau\delta$), the inequalities (41a) – (41d) hold for all μ when the exact dependence of p' on μ is taken into account. The total cross-section for scattering with spin-wave emission then becomes

$$\sigma^\tau_{+1} = \frac{S^2}{3}\, r_0^2\, \gamma^2 F^2(\tau)\, e^{-2W_\tau}\, \frac{g}{\tau p^2}\, \frac{(2\pi)^3}{v_0} \left[G(T) + \frac{1}{3}\right]. \tag{47}$$

From Eq. (10) we easily derive the total cross-sections for scattering with absorption or emission of one spin-wave, when p is large so that $\delta p \gg \sqrt{2\nu T/T_C}$, in a magnetized scatterer. We have only to multiply Eq. (44) or (47) by

$$^8/_3\,[3 - \cos^2\zeta - (\tau^2/4p^2)(1 - 3\cos^2\zeta)]. \tag{48}$$

We next examine the angular distribution of the scattered neutrons. The maximum intensity will be seen in the directions for which the energy of the absorbed or emitted spin-wave is a minimum. In particular, for $p \leq \tau/2$ the majority of neutrons will be scattered backward, while for $p > \tau/2$ the maximum scattered intensity will be at the Bragg angle θ_0. When p is large ($p\delta \gg \sqrt{2\nu T\alpha/T_C}$) so that $p' \approx p$, it is easy to compute the width of the diffuse maximum. In this case $\tau - \mu \leq q \leq \tau + \mu$ for scattering either with absorption or with emission. We find

$$\left|\sin\frac{\theta_0 \pm \Delta\theta}{2} - \sin\frac{\theta_0}{2}\right| \sim \frac{1}{2p\delta}\sqrt{\frac{2\nu T}{T_c}}. \tag{49}$$

For large p, we can also compute the differential cross-section,

$$d\sigma^\tau_{\pm 1}(p)/d\Omega = \int d\mu\, (d\sigma^\tau_{\pm 1}(p,\mu)/d\Omega).$$

for scattering with emission or absorption of a spin-wave. In this case

$$q \approx 2p\left(1 \pm \frac{\alpha\mu^2}{4p^2}\right)\sin\frac{\theta}{2}, \tag{50}$$

where the plus sign holds for absorption and the minus sign for emission. The limits of variation of q, together with Eq. (50), imply

$$|\mu_1^\pm(\theta)| \leqslant \mu \leqslant \mu_2^\pm(\theta), \tag{51}$$

$$\mu_1^{(\pm)}(\theta) = \frac{p}{\alpha\sin\frac{\theta}{2}}\left\{1 - \sqrt{1 \pm 4\alpha\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)}\right\}, \quad \mu_2^{(\pm)}(\theta) = \frac{p}{\alpha\sin\frac{\theta}{2}}\left\{1 + \sqrt{1 \pm 4\alpha\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)}\right\}, \tag{52}$$

where now the minus sign holds for absorption and the plus for emission. Further, we have

$$\sin\frac{\theta}{2} < \frac{1}{2}\sin\frac{\theta_0}{2}\left[1 + \sqrt{1 + \left(\alpha\sin^2\frac{\theta_0}{2}\right)^{-1}}\right] \tag{53}$$

for absorption, and

$$\sin\frac{\theta}{2} > \frac{1}{2}\sin\frac{\theta_0}{2}\left[1 + \sqrt{1 - \left(\alpha\sin^2\frac{\theta_0}{2}\right)^{-1}}\right] \tag{54}$$

for emission. In the region close to the Bragg maximum, we are concerned with spin-waves of small μ, and so unity may be neglected in comparison with $n(\mu)$ in Eq. (39). If we also neglect in Eq. (39) the variation of p' with μ, and set $q \approx \tau$, we obtain

$$d\sigma^{\tau}_{\pm 1}(p)/d\Omega \approx \frac{1}{8} S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau} (1+e_z^2) \frac{g_\tau}{\tau^2\delta^2} \frac{2\nu T}{T_c} \ln \frac{1-\exp\{-T_c[\mu_2^{(\pm)}(\theta)\delta]^2/2\nu T\}}{1-\exp\{-T_c[\mu_1^{(\pm)}(\theta)\delta]^2/2\nu T\}}, \qquad (55)$$

with e_z^2 given by Eq. (10).

The exponentials in Eq. (55) make the cross-section vanishingly small whenever the condition (49) is not satisfied. Very close to a Bragg peak, and at not too low a temperature, the cross-section is

$$\frac{d\sigma^{\tau}_{\pm}(p)}{d\Omega} \approx \frac{1}{4} S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau} (1+e_z^2) \frac{g_\tau}{(\tau\delta)^2} \frac{2\nu T}{T_c} \ln \frac{2p}{\alpha\tau\,|\theta-\theta_0|\cos(\theta_0/2)}. \qquad (56)$$

In conclusion I wish to thank A. I. Akhiezer for suggesting this investigation, and I. M. Shmushkevich and L. E. Gurevich for valuable criticism.

APPENDIX

Dyson[7,8] has shown that real spin-waves in a ferromagnet, although they have non-orthogonal wave-functions, correspond to "ideal" spin-waves in a fictitious model with orthogonal wave-functions. The probabilities of physical processes calculated for ideal spin-waves are equal to the probabilities of the same processes in the real physical system, provided that the temperature is low enough so that the concept of a spin-wave has a meaning. Also Avakiants[1] [see Eqs. (6) and (7)] has given a method of determining the operators of the ideal spin-wave model which correspond to real physical operators. Applying this method to the spin operators of atom number ℓ, we obtain

$$S_l^z = -S + \eta_l^+ \eta_l; \quad S_l^+ = \sqrt{2S}\,\eta_l^+; \quad S_l^- = \sqrt{2S}\left(1 - \frac{1}{2S}\eta_l^+\eta_l\right)\eta_l,$$

where the operators η_ℓ and η_ℓ^+ are connected with the spin-wave absorption and emission operators α_μ, α_μ^+ by the relations

$$\eta_l = N^{-1/2} \sum_\mu e^{i\mu R_l} \alpha_\mu; \quad \eta_l^+ = N^{-1/2} \sum_\mu e^{-i\mu R_l} \alpha_\mu^+.$$

These equations lead directly to Eq. (3).

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THE THEORY OF ELECTROMAGNETIC WAVES IN A CRYSTAL IN WHICH EXCITONS ARE PRODUCED

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Light waves at frequencies in the region of exciton absorption of light in a crystal are considered. A relationship is obtained between the specific dipole moment of polarization of the crystal and the electric field intensity. This relation is not a direct proportion, as customarily assumed, but is expressed by a differential equation. For this reason, the resultant Maxwell equations are of higher order and have more complicated solutions than the generally known solutions that follow from the simple phenomenological theory. It is shown that in a crystal there exist several waves of the same frequency, polarization, and propagation direction, but with different indices of refraction. This phenomenon differs from double refraction of light and occurs even in isotropically polarizing (cubic) crystals. Strictly-longitudinal electric waves are shown to exist in the crystal. Surface exciton states — analogues to surface electron states — are considered.

1. RELATION BETWEEN THE SPECIFIC DIPOLE MOMENT OF POLARIZATION AND THE ELECTRIC FIELD

LET Ψ^0 represent the ground state of the crystal, $\Psi_{\mathbf{k}}$ an excited exciton state with a quasi-momentum **k**, and Ψ_n, any other excited state of the crystal. By exciton state we mean here any excited state of the crystal, which has among its quantum numbers only one continuous (three-dimensional) quantum number **k**. Such a broad definition of an exciton embraces, for exmaple, Frenkel's exciton,[1] the Wannier-Mott exciton,[2,3] the simple phonon, or a molecular electron-oscillation excitation wave propagating through the crystal.

Further, let H be the energy operator for the crystal, and W operator for the interaction energy between the crystal and an applied electric field. The aforementioned wave functions and operators depend on the coordinates of the electron, as well as on those of the nuclei of the atoms in the crystal.

To investigate a crystal perturbed by a small alternating electric field, we write

$$\Psi = \Psi^0 + \psi + \psi', \ \psi = \sum_{\mathbf{k}} c_{\mathbf{k}} \Psi_{\mathbf{k}}, \ \psi' = \sum_n c_n \Psi_n, \tag{1}$$

wherein the entire time dependence is assumed to be contained in the coefficients $c_{\mathbf{k}}$ and c_n, so that $\Psi_{\mathbf{k}}$ and Ψ_n are independent of time. W, ψ, and ψ' will be considered quantities of the first order of magnitude. Then, retaining only first order terms in the Schrödinger time dependent equation, we obtain

$$\left[i\hbar \frac{\partial}{\partial t} - H\right] c_{\mathbf{k}} \Psi_{\mathbf{k}} = W_{\mathbf{k}0} \Psi_{\mathbf{k}}, \ W_{\mathbf{k}0} \equiv \int \Psi_{\mathbf{k}}^* W \Psi^0 d\Omega. \tag{2}$$

The integral is to be taken over the coordinates of all the particles of the crystal.

Exactly analogous equations are obtained for $c_n \Psi_n$.

Let us suppose that the electric field decreases quite slowly with distance (i.e., it changes only by a small amount over a distance on the order of the lattice constant). Then one can introduce the average specific dipole moment of the dielectric polarization of the crystal $\mathbf{P}(\mathbf{r})$, due to the external electric field

$$\mathbf{P}(\mathbf{r}) = \int \Psi^* \Psi \hat{\mathbf{P}}(\mathbf{r}) \, d\Omega, \quad d\Omega = \prod dx_i dy_i dz_i. \tag{3}$$

Here

$$\hat{\mathbf{P}}(\mathbf{r}) = \sum_i e_i (\mathbf{r}_i - \mathbf{r}) D(\mathbf{r}_i - \mathbf{r}), \tag{4}$$

where $D = 1$ for $-\frac{1}{2} \leq x, y, z \leq \frac{1}{2}$ and $D = 0$ for other values of x, y, and z. The summation is to be taken over all the particles comprising the crystal; i is the number of the particle, e is its charge, $\mathbf{r}_i$ is its coordinate and $\mathbf{r}$ is the coordinate of the macroscropic point in the crystal, with relation to which the value of the specific dipole moment is determined.

Retaining only first-order terms, we obtain

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_1(\mathbf{r}) + \mathbf{P}_2(\mathbf{r}) + \mathbf{P}'(\mathbf{r}),\ \mathbf{P}_1(\mathbf{r}) \equiv \int \Psi^{0*}\psi \hat{\mathbf{P}}(\mathbf{r})\, d\Omega,\ \ \mathbf{P}_2(\mathbf{r}) \equiv \int \psi^{*}\Psi^{0} \hat{\mathbf{P}}(\mathbf{r})\, d\Omega,\ \ \mathbf{P}'(\mathbf{r}) \equiv \int (\Psi^{0*}\psi' + \psi'^{*}\Psi^{0}) \hat{\mathbf{P}}(\mathbf{r})\, d\Omega. \tag{5}$$

Let $T_{\mathbf{m}}$ represent the translation operation for all the electrons and for all deformations of the lattice by an integral lattice vector $\mathbf{m} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$ ($\mathbf{a}_j$ are the lattice constants). In other words, $T_{\mathbf{m}}$ represents a change in the configuration of the crystal, such that every electron is displaced by the vector $\mathbf{m}$ $(\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{m})$, and every atom (nucleus) of type ℓ in the elementary cell number $\mathbf{n}$ is displaced to the site formerly occupied by the atom of the same type located in cell number $\mathbf{n} - \mathbf{m}\, \{\mathbf{r}^{\ell}_{\mathbf{n}} - \mathbf{n} \rightarrow \mathbf{r}^{\ell}_{\mathbf{n}-\mathbf{m}} - (\mathbf{n} - \mathbf{m})\}$. As a result of such a coordinate transformation, H remains invariant. Therefore, making use of the well known methods of group theory, one can obtain the following results: the system of eigenfunctions of the operator H can always be chosen such that they posses the property:

$$T_{\mathbf{m}}\Psi_{\mathbf{f}} = e^{i\mathbf{f}\mathbf{m}}\Psi_{\mathbf{f}}, \tag{6}$$

where

$$\mathbf{f} = g_1\mathbf{b}_1 + g_2\mathbf{b}_2 + g_3\mathbf{b}_3,\quad -\pi \leqslant g_1, g_2, g_3 \leqslant \pi,$$

where $\mathbf{b}_i$ are the reciprocal lattice constants. The ground state of the crystal Ψ^0, as is well known, must be real and positive. Therefore $T_{\mathbf{m}}\Psi^0 = \Psi^0$. An exciton state must satisfy the relation (6), in which the quantum number $\mathbf{f}$ should be identified with $\mathbf{k}$, for, by definition, an exciton state is characterized generally by one (three-dimensional) continuous quantum number $\mathbf{k}$. Examination of concrete models of excitons shows that $\Psi_{\mathbf{k}}$ and the energy of the crystal $\mathscr{E}(\mathbf{k})$ turn out to be continuous functions of $\mathbf{k}$, i.e., of $\mathbf{f}$, in the region denoted by Eq. (6).

Furthermore, from Eq. (4) it follows immediately that the specific dipole moment $\hat{\mathbf{P}}$ at the point $\mathbf{r}$ acquires after the translation the value that it had prior to the translation to the point $\mathbf{r} - \mathbf{m}$: $T_{\mathbf{m}}\hat{\mathbf{P}}(\mathbf{r}) = \hat{\mathbf{P}}(\mathbf{r} - \mathbf{m})$. By a simple substitution of the variables of integration it can be readily shown that

$$\int T_{\mathbf{m}} F(\cdots \mathbf{r}_i \cdots)\, d\Omega = \int F(\cdots \mathbf{r}_i \cdots)\, d\Omega.$$

Using the above translation properties of the functions, we obtain

$$\mathbf{P}_{0\mathbf{k}}(\mathbf{r}) \equiv \int \Psi^{0*}\hat{\mathbf{P}}(\mathbf{r})\Psi_{\mathbf{k}}\, d\Omega = \int T_{\mathbf{m}}[\Psi^{0*}\hat{\mathbf{P}}(\mathbf{r})\Psi_{k}]\, d\Omega = e^{i\mathbf{k}\mathbf{m}}\mathbf{P}_{0\mathbf{k}}(\mathbf{r} - \mathbf{m}). \tag{7}$$

Since an external electric field is a smooth function of the coordinates, then, as will be shown below, the main contribution will be made by terms with small values of $\mathbf{k}$. In such a case, the quantity (7) changes only slightly over distances on the order of the lattice constant and one can write in (7) that $\mathbf{m} = \mathbf{r}$ approximately. Then

$$\mathbf{P}_{0\mathbf{k}}(\mathbf{r}) = \mathbf{P}_0 e^{i\mathbf{k}\mathbf{r}},\quad \mathbf{P}_0 = \mathbf{P}_{0\mathbf{k}}(0) \approx [\mathbf{P}_{0\mathbf{k}}(0)]_{\mathbf{k}=0}. \tag{8}$$

Let us now turn to the calculation of $\mathbf{P}_1(\mathbf{r})$ in (5). In accordance with Eq. (1)

$$\mathbf{P}_1(\mathbf{r}) = \sum_{\mathbf{k}} c_{\mathbf{k}} \int \Psi^{0*}\Psi_{\mathbf{k}}\hat{\mathbf{P}}(\mathbf{r})\, d\Omega = \sum_{\mathbf{k}} c_{\mathbf{k}}\mathbf{P}_{0\mathbf{k}}(\mathbf{r}). \tag{9}$$

For simplicity one can choose the energy scale such that the energy of the ground state of the crystal is zero. Then Ψ^0 is time-independent, and on the basis of (8) and (2) one can write

$$\left[i\hbar\frac{\partial}{\partial t} - \mathscr{E}(-i\nabla)\right] c_{\mathbf{k}}\mathbf{P}_{0\mathbf{k}} = \left[i\hbar\frac{\partial}{\partial t} - \mathscr{E}(\mathbf{k})\right] c_{\mathbf{k}}\mathbf{P}_{0\mathbf{k}} = \int \Psi^{0*}\hat{\mathbf{P}}\left[i\hbar\frac{\partial}{\partial t} - H\right] c_{\mathbf{k}}\Psi_{\mathbf{k}}\, d\Omega = \mathbf{P}_{0\mathbf{k}}W_{\mathbf{k}0}, \tag{10}$$

where $\mathscr{E}(\mathbf{k})$ is the energy of the exciton state.

Summing this expression over $\mathbf{k}$ and taking Eq. (9) into account, we obtain

$$[i\hbar\partial/\partial t - \mathscr{E}(-i\nabla)]\,\mathbf{P}_1(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{P}_{0\mathbf{k}}(\mathbf{r})\,W_{\mathbf{k}0}. \tag{11}$$

The energy of the interaction between the external electromagnetic field and the crystal can be written in the form

$$W = -\frac{1}{c}\int \dot{\hat{\mathbf{P}}}(\mathbf{r}')\,\mathbf{A}(\mathbf{r}', t)\,d\tau', \quad W_{\mathbf{k}0} = -\frac{1}{c}\int \dot{\mathbf{P}}_{\mathbf{k}0}(\mathbf{r}')\,\mathbf{A}(\mathbf{r}', t)\,d\tau' = -\frac{i\omega_0\mathbf{P}_0^*}{c}\int e^{-i\mathbf{k}\mathbf{r}'}\mathbf{A}(\mathbf{r}', t)\,d\tau', \tag{12}$$

where $\mathbf{A}$ is the vector potential of the field, and $\omega_0 = \mathscr{E}(0)/\hbar$. Here it is seen that with increasing $\mathbf{k}$, $W_{\mathbf{k}0}$ rapidly approaches zero, as a coefficient of the Fourier expansion of a smooth function of the coordinates $\mathbf{A}$. Therefore, in the summation on the right side of Eq. (11), terms with small values of $\mathbf{k}$ dominate, and one can approximately extend the summation over the entire infinite $\mathbf{k}$ space. Then, substituting the results of Eqs. (8) and (12) into (11), and also taking it in account that

$$\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} = V\delta(\mathbf{r}-\mathbf{r}')$$

(V is the volume of the crystal) we obtain finally

$$[i\hbar\partial/\partial t - \mathscr{E}(-i\nabla)]\,\mathbf{P}_1(\mathbf{r}, t) = -(i\omega_0 V/c)\,\mathbf{P}_0(\mathbf{P}_0^*, \mathbf{A}(\mathbf{r}, t)). \tag{13}$$

Differentiation of this equation with respect to time yields

$$[i\hbar\partial/\partial t - \mathscr{E}(-i\nabla)]\,\dot{\mathbf{P}}_1(\mathbf{r}, t) = ia\mathbf{E}_\perp(\mathbf{r}, t), \quad a_{xy} = \omega_0 V P_{0x} P_{0y}^*, \tag{14}$$

where $\mathbf{E}_\perp = \dot{\mathbf{A}}/c$ is the vortical part of the electric field intensity ($\operatorname{div}\mathbf{E}_\perp = 0$) and a is a tensor of second rank.

The equation for the term $\mathbf{P}_2(\mathbf{r}, t)$, which appears in Eq. (5), is most readily obtained from Eq. (14), if one notes that $\mathbf{P}_2 = \mathbf{P}_1^*$:

$$[i\hbar\partial/\partial t + \mathscr{E}(-i\nabla)]\,\dot{\mathbf{P}}_2(\mathbf{r}, t) = ia^*\mathbf{E}_\perp(\mathbf{r}, t). \tag{15}$$

In most cases $\mathscr{E}(\mathbf{k})$ for small $\mathbf{k}$ has the form

$$\mathscr{E}(\mathbf{k}) = \mathscr{E}_0 + \frac{\hbar^2}{2}\sum_{ps} M_{ps}^{-1} k_p k_s + \ldots; \quad p, s = 1, 2, 3; \quad \mathscr{E}_0 = \mathscr{E}(0). \tag{16}$$

Here M^{-1} is the tensor of the reciprocal effective mass of the exciton. Then Eq. (14) can be rewritten

$$\left[i\hbar\frac{\partial}{\partial t} - \mathscr{E}_0 + \frac{\hbar^2}{2}\sum_{ps} M_{ps}^{-1}\frac{\partial^2}{\partial x_p \partial x_s}\right]\dot{\mathbf{P}}_1 = ia\mathbf{E}_\perp \tag{17}$$

Thus, in a dielectric in which excitons can be generated, the relation between the specific dipole moment and the electric field intensity is given not by the generally well-known "material" Maxwell equations, but rather by the differential equations (17). The latter must be supplemented with initial conditions, i.e., we must specify $\mathbf{P}_1$ and $\dot{\mathbf{P}}_1$ at $t = 0$, and also with boundary conditions, i.e., we must specify the value of $\mathbf{P}_1$, say, on the surface of the crystal. Subsequent sections of the paper are devoted to a formulation of the boundary conditions.

The term $\mathbf{P}'$ in (5) is associated with virtual transitions into non-exciton states Ψ_n. It will not be investigated in detail in this paper, and presumably this term can be taken into account by introducing a conventional phenomenological dielectric constant ϵ' (which is, in general, complex)

$$\mathbf{P}' = (\varepsilon' - 1)\,\mathbf{E}/4\pi, \tag{18}$$

where $\mathbf{E}$ is the electric field intensity. In reality this relation is not always satisfied. This question requires a special investigation.

The results described above were obtained under the assumption that when the crystal is unperturbed by light it remains in its ground state Ψ^0, and as a result of illumination a slight admixture of excitons, i.e., of states with the same momentum $\Psi_{\mathbf{k}}$, is added to Ψ^0. This assumption is strictly satisfied only at absolute zero. At other temperatures there exist in the crystal many phonons, both in the absence and in the presence of the perturbation by the light. In reality, however, these results will hold approximately for temperatures other than zero, provided that they are still sufficiently low.

2. EFFECT OF THE CRYSTAL SURFACE ON EXCITON STATES; BOUNDARY CONDITIONS ON THE SPECIFIC POLARIZATION DIPOLE MOMENT. SURFACE EXCITONS

To consider the boundary conditions at the crystal surface, it is necessary to make the exciton model more concrete. Let us consider Frenkel's[1] exciton: let us assume that the interaction between the crystal molecules is quite weak and that it is therefore possible to construct the many-electron wave function of the crystal, in the zeroth approximation, according to the method of Heitler-London-Heisenberg (HLH). Let $\Phi_\mathbf{n}$ represent the (antisymmetrized) wave function of the crystal, when the electron-oscillation excitation is localized in the $\mathbf{n}$-th elementary cell. Since the excitation can be localized in any cell with equal success and since all such states $\Phi_\mathbf{n}$ with the same $\mathbf{n}$ are mutually degenerate, then in the zeroth approximation the crystal wave function should be constructed as a linear combination of all $\Phi_\mathbf{n}$. One can readily verify that

$$T_\mathbf{m}\Phi_\mathbf{m} = \Phi_0,\ \Phi_\mathbf{m} = T_{-\mathbf{m}}\Phi_0. \tag{19}$$

Here Φ_0 is the state of the crystal, when the excited cell coincides with the origin of coordinates. If we specify that the above-mentioned linear combination possess the translation properties of Eq. (6), then the coefficients of the linear combination are uniquely determined from Eqs. (6) and (19). Consequently, the wave function in the zeroth approximation has the form

$$\Psi_\mathbf{k} = \sum_\mathbf{n} a_\mathbf{n}\Phi_\mathbf{n},\ a_\mathbf{n} = a_0 e^{i\mathbf{kn}}. \tag{20}$$

This same zeroth-approximation function is obtained, if instead of the methods of group theory one employs the conventional perturbation method for excited states, by taking into account the interaction between neighboring elementary cells of the crystal as small perturbations. As a result, one obtains for $a_\mathbf{n}$ the following system of homogeneous linear equations:

$$(\mathscr{E} - H_0)\, a_\mathbf{n} = \sum_{i=1}^{3} H_i\, (a_{\mathbf{n}+\mathbf{a}_i} + a_{\mathbf{n}-\mathbf{a}_i}), \tag{21}$$

where $\mathbf{a}_i$ are lattice constants, and

$$H_0 = \int \Phi_0 H \Phi_0 d\Omega,\ H_i = H_{0\mathbf{a}_i} = H_{\mathbf{a}_i 0} = \int \Phi_{\mathbf{a}_i} H \Phi_0 d\Omega.$$

One can easily show that the values of $a_\mathbf{n}$ from Eq. (20) are solutions of Eq. (21). From this one obtains for the crystal energy the value[1]

$$\mathscr{E}(\mathbf{k}) = H_0 + 2\sum_{i=1}^{3} H_i \cos g_i,\quad \mathbf{k} = g_1\mathbf{b}_1 + g_2\mathbf{b}_2 + g_3\mathbf{b}_3. \tag{22}$$

The solutions (20) and (22), as well as the results of the preceding section are applicable to an infinite crystal. In addition, in obtaining Eqs. (21) and (22) it was assumed that every cell in the crystal had six adjacent cells.

Now let us turn to the consideration of a finite crystal. Let it have the form of an infinite plane — parallel lamina bounded by the surfaces $n_3 = 1$ and $n_3 = G$ (G is a large integer). Thus, junctions between crystal cells occur only at points with vectors $\mathbf{n}$, whose third component is included in the interval $1 \leq n_3 \leq G$. If now one sets up the system of equations for $a_\mathbf{n}$, then all those equations from the set of Eqs. (21), in which the vector $\mathbf{n}$ does not belong to surface cells of the crystal, will remain unchanged. At this surface itself Eqs. (21) are changed as follows: for $n_3 = 1$ the term $a_{\mathbf{n}-\mathbf{a}_3}$ of the right side is simply absent, and for $n_3 = G$ the term $a_{\mathbf{n}+\mathbf{a}_3}$ of the right side is absent. In solving such a system of equations it is helpful to keep in mind that the translation symmetry in the direction $\mathbf{a}_3$ is now absent, whereas in the directions $\mathbf{a}_1$ and $\mathbf{a}_2$ it is still there as before. Therefore the wave function should satisfy the relation (6), in which one should put $m_3 = 0$. Finally one obtains

$$a_\mathbf{n} = \frac{a_0}{\sqrt{2}}\,[e^{i\mathbf{kn}} - e^{i\tilde{\mathbf{k}}\mathbf{n}}],\ g_3 = \pi\frac{\nu}{G+1},\ \nu = 1, 2, 3 \ldots G. \tag{23}$$

Here the vector $\tilde{\mathbf{k}}$ represents the specular reflection of the vector $\mathbf{k}$ in the surface plane of the crystal. In other words, $\tilde{\mathbf{k}}$ is uniquely determined from $\mathbf{k}$ by the relations $\tilde{\mathbf{k}} \cdot \mathbf{a}_1 = \mathbf{k} \cdot \mathbf{a}_1$, $\tilde{\mathbf{k}} \cdot \mathbf{a}_2 = \mathbf{k} \cdot \mathbf{a}_2$, and $\tilde{\mathbf{k}} \cdot \mathbf{a}_3 = -\mathbf{k} \cdot \mathbf{a}_3$. In accordance with Eqs. (23) and (20), the exciton wave function in a bounded crystal can be written in the form:

$$\Phi_{\mathbf{k}} = (\Psi_{\mathbf{k}} - \Psi_{\tilde{\mathbf{k}}})/\sqrt{2}. \tag{24}$$

If one denotes the exciton wave incident on the surface by $\Psi_{\mathbf{k}}$, then the wave which is specularly reflected from the surface of the crystal is denoted by $\Psi_{\tilde{\mathbf{k}}}$. In Eq. (24) $\Psi_{\mathbf{k}}$ denotes, of course not the wave function of an infinite crystal, but rather the wave function of the lamina described above, under the condition of periodicity in the direction $\mathbf{a}_3$ with a period $G\mathbf{a}_3$. As a result, $\Psi_{\mathbf{k}}$ also possesses the translation properties of Eq. (6), also in direction $\mathbf{a}_3$. The dependence of the energy on $\mathbf{k}$ is given by Eq. (22) as before.

It is important to emphasize, that the results of the preceding section can be applied also to the case of a bounded crystal. To do so one must change $\Psi_{\mathbf{k}}$ to $\Phi_{\mathbf{k}}$ in Eqs. (1), (2), (9), and (11). Equations (13) through (17) remain precisely the same. Only Eq. (8) is changed and becomes for the bounded crystal

$$\mathbf{P}_{0\mathbf{k}}(\mathbf{r}) = \mathbf{P}_0 (e^{i\mathbf{kr}} - e^{i\tilde{\mathbf{k}}\mathbf{r}})/\sqrt{2}, \tag{25}$$

in which the constant $\mathbf{P}_0$ has the previous numerical value.

Now it is easy to obtain the value of $\mathbf{P}(\mathbf{r})$ at the crystal surface. One of the surfaces of the crystal ($n_3 = 1$) in the macroscopic sense is the plane passing through the origin of coordinates. For points on this surface $\mathbf{r}$ is coplanar with $\mathbf{a}_1$ and $\mathbf{a}_2$ and it can be represented thus: $\mathbf{r} = \mathbf{a}_1 r_1 + \mathbf{a}_2 r_2$. Under these conditions the difference in the indices of the exponentials in Eq. (25) becomes

$$(\mathbf{k} - \tilde{\mathbf{k}}, \mathbf{r}) = (\mathbf{k} - \tilde{\mathbf{k}}, \mathbf{a}_1) r_1 + (\mathbf{k} - \tilde{\mathbf{k}}, \mathbf{a}_2) r_2 = 0,$$

in accordance with the definition of the vector $\tilde{\mathbf{k}}$. Therefore, on this plane $\mathbf{P}_{0\mathbf{k}}(\mathbf{r}) = 0$. The other surface of the lamina ($n_3 = G$) consists of points whose coordinates are $\mathbf{r} = \mathbf{a}_1 r_1 + \mathbf{a}_2 r_2 + \mathbf{a}_3 (G + 1)$. In this case the difference in the exponents of Eq. (25) is [see Eq. (23)]

$$(\mathbf{k} - \tilde{\mathbf{k}}, \mathbf{r}) = (\mathbf{k} - \tilde{\mathbf{k}}, \mathbf{a}_3)(G + 1) = 2(\mathbf{k}\mathbf{a}_3)(G + 1) = 2\pi\nu.$$

Consequently $\mathbf{P}(\mathbf{r}) = 0$ on this surface, too. Then Eq. (9) shows that $\mathbf{P}_1(\mathbf{r}) = 0$ on the surfaces of the crystal. Thus the surfaces of the crystal turn out to be nodal surfaces for exciton waves as well as for the specific dipole moment.

As is well known, in a bounded crystal conduction electrons can exist in surface states as well as in "band" states.[4] In an exactly analogous way, surface exciton states can exist, in which the excitation wave moves only in the layer near the surface of the crystal. In order to obtain these surface excitons in the HLH approximation, let us consider a semi-infinite crystal, whose lattice points are located in the region $n_3 \geq 1$. As before, Eq. (21) is satisfied for all $\mathbf{n}$ not associated with cells adjoining the surface (i.e., for $n_3 \geq 2$). The solution of these equations, just as in the case of the infinite crystal, can be cast in the form of Eqs. (20) and (22). Since the translation symmetry in the directions $\mathbf{a}_1$ and $\mathbf{a}_2$ is strictly conserved, g_1 and g_2, as before, must be real and contained within the interval specified by Eq. (22). But now g_3 can also assume complex values, for Eqs. (21) satisfy Eq. (20) for any complex $\mathbf{k}$, the energy [Eq. (22)] obtained is real, and the wave function remains finite, provided that one puts $g_3 = i\kappa + \ell\pi$ for $\kappa \geq 0$ and ℓ an integer. Thus, Eqs. (21) also have a solution of the form

$$a_{\mathbf{n}} = (-1)^l a_0 \exp\{i(g_1 n_1 + g_2 n_2) - \varkappa n_3\}, \quad -\pi \leqslant g_1,\ g_2 \leqslant \pi,\ \varkappa \geqslant 0; \tag{26}$$

$$\mathscr{E}(\mathbf{k}) \equiv \mathscr{E}(g_1,\ g_2,\ g_3) = H_0 + 2\sum_{i=1}^{2} H_i \cos g_i + 2(-1)^l H_3 \cosh \varkappa. \tag{27}$$

Now it is still necessary to satisfy the equation for $a_{\mathbf{n}}$ when $\mathbf{n}$ pertains to cells near the surface of the crystal. These equations differ from (21) in two particulars: (a) for $n_3 = 1$ in the right side of the equation there exists a term $H_3 a_{\mathbf{n}-\mathbf{a}_3}$, (b) the constants H_0 and H_i have a somewhat different value, because of the absence of an adjoining layer $n_3 = 0$. The coefficients, H_i, in the HLH approximation are first-order quantities while their changes near the surface of the crystal are second-order quantities and will therefore be neglected. H_0 is a zeroth order term, and its change near the surface H_0' is a first-order term, and must be allowed for. As a result, when n pertains to surface cells ($n_3 = 1$), Eq. (21) is replaced by

$$(\mathscr{E} - H_0 - H_0')\, a_{\mathbf{n}} = \sum_{i=1}^{3} H_i\, [a_{\mathbf{n}+\mathbf{a}_i} + a_{\mathbf{n}-\mathbf{a}_i} (1 - \delta_{i3})]. \tag{28}$$

We now require that the solutions of the "depth" equations (21), given by (26) and (27), also satisfy the "subsurface" equations (28). This leads to a relation for κ:

$$e^{-\varkappa} = (-1)^l H_3 / H_0'. \tag{29}$$

If a positive value of κ results from this expression, i.e., if $|H_3/H_0'| \le 1$, then Eqs. (26), (27), and (29) are the solutions of the problem of the semi-infinite crystal. From this one finds that the probability that the excitation will remain in a cell with coordinate n is given by $|a_{\mathbf{n}}|^2$. This probability does not depend on n_1 and n_2 and decreases exponentially with increasing n_3, i.e., with increasing distance from the crystal surface. The energy of the system, Eq. (27) represents a two dimensional band, because the projection of the vector **k** on the direction $\mathbf{a}_3$ remains fixed. For convenience, such states are called surface excitons. If $|H_3/H_0'| > 1$, no surface excitons exist.

The results of the previous section can be extended to include surface excitons. Equations (1) through (5) remain valid even in this case provided that $\Psi_{\mathbf{k}}$ is taken to mean the surface exciton state. Now the validity of Eq. (6) cannot be proven for any arbitrary translation $T_{\mathbf{m}}$ by the methods of group theory, since there is no translational symmetry in the direction $\mathbf{a}_3$. Nevertheless, one can substantiate it as follows. Let us impose a periodicity with a large period $G\mathbf{a}_3$ on the direction $\mathbf{a}_3$ (as we did above when we were considering the plane-parallel lamina. Then Eqs. (19) and (20) yield

$$T_{\mathbf{m}}\Psi_{\mathbf{k}} = \sum_{\mathbf{n}} a_{\mathbf{n}} T_{\mathbf{m}} T_{-\mathbf{n}} \Phi_0 = \sum_{\mathbf{n}} a_{\mathbf{n}} T_{\mathbf{m}-\mathbf{n}} \Phi_0 = \sum_{\mathbf{n}} a_{\mathbf{n}+\mathbf{m}} T_{-\mathbf{n}} \Phi_0 = e^{i\mathbf{k}\mathbf{m}} \Psi_{\mathbf{k}}, \tag{30}$$

where this relation holds for both real and complex **k**. In this way, Eqs. (6) through (12) remain valid for surface excitons, but one must remember that the values of **k** are restricted by the inequalities (26) and the conditions (29). In Eq. (8) one must now put $\mathbf{P}_0 = \mathbf{P}_{0\mathbf{k}}(0)$ with $g_1 = g_2 = 0$; $g_3 = i\kappa$, and in Eq. (12) one must integrate only over the half-space occupied by the crystal.

Substituting Eqs. (8) and (12) into Eq. (11) and noting that

$$\sum_{g_1, g_2} e^{i(\mathbf{k}\mathbf{r} - \mathbf{k}^*\mathbf{r}')} = G^2 e^{-\varkappa(r_3 + r_3')}\, \delta(r_1 - r_1')\, \delta(r_2 - r_2'), \tag{31}$$

where the new variables r_1, r_2, and r_3 are defined by the relation $\mathbf{r} = \mathbf{a}_1 r_1 + \mathbf{a}_2 r_2 + \mathbf{a}_3 r_3$, we obtain for surface excitons

$$\mathbf{P}_1 = \mathbf{P}_1^{(s)}(r_1, r_2)\, e^{-\varkappa r_3},$$

$$\left[i\hbar \frac{\partial}{\partial t} - \mathscr{E}\left(-i \frac{\partial}{\partial r_1}, -i \frac{\partial}{\partial r_2}, i\varkappa\right)\right] \mathbf{P}_1^{(s)} = -\frac{i\omega_0}{c} G^2 v \mathbf{P}_0 \int_0^{\infty} e^{-\varkappa r_3'} (\mathbf{P}_0^*, \mathbf{A}(\mathbf{a}_1 r_1 + \mathbf{a}_2 r_2 + \mathbf{a}_3 r_3', t))\, dr_3', \tag{32}$$

Here $v = \mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$, the volume of the elementary cell of the crystal, and $\mathscr{E}(g_1, g_2, g_3)$ is defined by Eq. (27). If $\mathbf{A}(\mathbf{r}, t)$, as a function of **r**, does not change very much over distances of the order of $1/\kappa$, one can move **A** in Eq. (32) outside the integration sign, since it is a smooth function. Finally, differentiation of Eq. (32) with respect to time yields

$$\left[i\hbar \frac{\partial}{\partial t} - \mathscr{E}\left(-i \frac{\partial}{\partial r_1}, -i \frac{\partial}{\partial r_2}, i\varkappa\right)\right] \dot{\mathbf{P}}_1^{(s)} = i a^{(s)} \mathbf{E}_{\perp}(\mathbf{a}_1 r_1 + \mathbf{a}_2 r_2, t), \quad a_{xy}^{(s)} = \frac{\omega_0 G^2 v}{\varkappa} P_{0x} P_{0y}^*. \tag{33}$$

This is the analogue of Eq. (14) for the case of surface excitons.

Within the framework of the HLH approximation one can relate the constants a_{xx} and a_{xx}^{S} to the oscillator strength of a given transition f_x per unit cell of the crystal. For example, for excitons associated with electron excitation one obtains

$$a_{xx} = (e^2\hbar/2m) N f_x, \quad a_{xx}^{(s)} = 2a_{xx}, \tag{34}$$

where N is the number of cells per unit volume and m is the free-electron mass.

In the discussion above we have considered the effect of the surface on the exciton, for which we used Frenkel's model of the exciton. Similar results, however, are obtained for other models of the exciton. For example, in the case of the Wannier-Mott[2,3] exciton, in which the conduction electron and the hole are bound together by Coulomb attraction and perform hydrogen-like motions, it is easily shown that the exciton wave is specularly reflected by the surface. For a plane-parallel lamina one again obtains Eqs. (24) and (25), and g_3 assumes a discrete series of values, as indicated in Eq. (23). Therefore, just as in the case considered above, the surface of the crystal is a nodal surface for exciton waves as well as for the specific dipole moment $\mathbf{P}_1(\mathbf{r}, t)$.

Surface exciton states can also exist for electron-hole excitons. They arise for example, if a conduction hole becomes bound to an electron, located in a surface conduction band, or if, conversely, an electron becomes bound to a hole which is in a surface state. It is also possible for an electron and hole to become coupled, both of them belonging to a surface band. For this case one obtains (32) and (33) again.

If the exciton is an ordinary acoustical phonon, then as is well known for the case of an elastic wave, the displacement vector of the medium does not have a node at the surface, but rather an antinode (the vacuum bounds the surface). However, the specific dipole moment is proportional not to the displacement but rather to the derivative of the displacement with respect to the coordinates, and has therefore, as before, a node at the surface. Surface excitons, in the case of phonons, are Rayleigh surface waves, which have been known for a long time.

In the various models of the bulk exciton considered above, it was found that at the surface of the crystal $\mathbf{P}_1 = 0$, which is a boundary condition for Eq. (14). Now and then, deviations from this rule are observed. For example, in the case of phonons, if the surface of the crystal is fixed (the crystal borders on a more rigid or a heavier medium) the displacements have a node at the surface and the specific polarization, correspondingly, has an antinode.

For surface excitons, one does not obtain any boundary conditions at the surface of the crystal. Equations (32) and (33) are not required for them. For the variables r_1 and r_2 the boundary conditions reduce to conditions of periodicity.

3. PLANE ELECTROMAGNETIC WAVES IN A CRYSTAL

In this section, we shall solve Maxwell's equations for the case when the relation between the specific polarization and the applied electric field has the form of Eq. (16) or Eq. (17). Maxwell's equations can be divided into two groups: the "ideal" equations which are independent of the properties of the crystal

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}},\quad \operatorname{curl}\mathbf{H} = \frac{1}{c}\dot{\mathbf{D}},\quad \operatorname{div}\mathbf{D} = 0,\quad \operatorname{div}\mathbf{H} = 0, \tag{35}$$

and the "material" equations

$$\mathbf{D} = \varepsilon'\mathbf{E} + 4\pi(\mathbf{P}_1 + \mathbf{P}_2),\quad [i\hbar\,\partial/\partial t - \mathcal{E}(-i\nabla)]\,\dot{\mathbf{P}}_1 = ia\mathbf{E}_\perp,\quad [i\hbar\partial/\partial t + \mathcal{E}(-i\nabla)]\,\dot{\mathbf{P}}_2 = ia^*\mathbf{E}_\perp. \tag{36}$$

We shall now look for solutions of these equations in the form of plane waves, thus obtaining $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{P}_1$, and $\mathbf{P}_2$ proportional to $\exp\{i\omega[n(\mathbf{r}\cdot\mathbf{s})/c - t]\}$. Here ω is the frequency of the wave, n is the index of refraction, $\mathbf{s}$ is a unit vector, normal to a constant phase surface. As a result, it is found that Eqs. (35) are satisfied if the amplitudes of the fields are related as follows

$$n[\mathbf{s}\times\mathbf{E}] = \mathbf{H},\quad n[\mathbf{s}\times\mathbf{H}] = -\mathbf{D}. \tag{37}$$

If $\mathbf{H}$ is eliminated from these equations, one obtains

$$\mathbf{D} = n^2[\mathbf{E} - \mathbf{s}(\mathbf{s}\mathbf{E})]. \tag{38}$$

Not only Eqs. (37) and (38), but also other relations of ordinary crystal optics, following from the "ideal" equations, remain valid. These relations are to be found, for example, in Born's book[5] (pp. 296–299).

Substituting the exponential solutions into the second and third of Eqs. (36) yields

$$\mathbf{P}_1 = \frac{a}{\omega}\,\frac{\mathbf{E}_\perp}{\mathcal{E}(\omega n\mathbf{s}/c) - \hbar\omega},\quad \mathbf{P}_2 = \frac{a^*}{\omega}\,\frac{\mathbf{E}_\perp}{\mathcal{E}(\omega n\mathbf{s}/c) + \hbar\omega},\quad \mathbf{E}_\perp = \mathbf{E} - \mathbf{s}(\mathbf{s}, \mathbf{E}) \equiv \eta\mathbf{E}, \tag{39}$$

where η is a tensor of the second rank: $\eta_{xy} = 1 - s_x s_y$. Here we can use for $\mathcal{E}$ the quadratic expansion

$$\mathscr{E}(\omega n s/c) = \mathscr{E}_0 + \hbar^2\omega^2 n^2/2M_s c^2 + \cdots, \tag{40}$$

since it is assumed that the wavelength of the light is much greater than the lattice constant. M_s depend on the direction of s and we shall henceforth refer to it as the effictive mass of the exciton in the direction s. M_s can be easily related to the tensor of the reciprocal effective mass, calculated in Eq. (17). In Eq. (39) $\mathbf{P}_2 \neq \mathbf{P}_1^*$, but if subsequently one constructs a linear-combination solution in which the condition that the field is real if preserved (this is done by adding together exponentials with different signs i front of ω), then one obtains $\mathbf{P}_2 = \mathbf{P}_1^*$.

If the frequency of the light wave, ω, lies outside the region of absorption by excitons, i.e., if it differ appreciably from $\omega_0 = \mathscr{E}_0/\hbar$ then one can replace $\mathscr{E}(\omega \mathbf{n}\mathbf{s}/c)$ by $\mathscr{E}_0$ in Eq. (39). In this case, the "material" equations of Maxwell preserve their customary well known form, and one obtains the results of ordinary crystal optics. But in the vicinity of the region of exciton absorption such a substitution can be made only in the expression for $\mathbf{P}_2$; in the expression for $\mathbf{P}_1$ one must take into account the functional dependence of $\mathscr{E}$ on n. As a result one can neglect $\mathbf{P}_2$ in comparison with $\mathbf{P}_1$ and the "material" Eqs. (36) can be formally written in the customary way $\mathbf{D} = \epsilon\mathbf{E}$. However, the tensor ϵ is now given by

$$\varepsilon = \varepsilon' + B\eta/(n^2 - \mu), \tag{41}$$

where

$$B = 8\pi M_s c^2 a/\hbar^2\omega^3, \qquad \mu = (2M_s c^2/\hbar\omega)(1 - \mathscr{E}_0/\hbar\omega).$$

Thus, ϵ is now a function not only of the frequency ω, but also of s and of n. Consequently, to every direction of propagation of the light wave and every direction of its polarization there corresponds a specific ϵ tensor with its own direction for the principal axes of polarization.

Eliminating D from the material equations and from Eq. (38), we finally obtain

$$\varepsilon\mathbf{E} = n^2[\mathbf{E} - \mathbf{s}(\mathbf{s}\mathbf{E})]. \tag{42}$$

Equations (41) and (42) are equivalent to the initial Maxwell equations, Eqs. (35) and (36), if the solution is given in the form of plane exponential waves.

Equations (42) represent a system of three linear, single-valued equations for the components of the amplitude of the electric field. Setting the determinant of this system equal to zero, we obtain the equations for n:

$$(\mathbf{s},\varepsilon\mathbf{s})\,n^4 + [(\mathbf{s},\varepsilon^2\mathbf{s}) - (\mathbf{s},\varepsilon\mathbf{s})\,\mathrm{Sp}\,\varepsilon]\,n^2 + \Delta(\varepsilon) = 0, \tag{43}$$

where $\Delta(\epsilon)$ is the determinant of the tensor ϵ. In the well-known ordinary crystal optics, ϵ is not a function of n, and Eq. (43) is quadratic in n^2. The two roots of this equation determine for a given s two values of |n|, and accordingly two plane waves, for which the D vectors have mutually perpendicular directions. However, it was shown above that near the exciton absorption region, when μ is not large, it is necessary to take into account the dependence of ϵ on n, in accordance with Eq. (41). If (41) is substituted into (43) and for simplification we introduce the symbols

$$\overline{\varepsilon'} = (\mathbf{s},\ \varepsilon'\mathbf{s}),\ \overline{\varepsilon'^2} = (\mathbf{s},\varepsilon'^2\mathbf{s}),\ \overline{B\eta} = (\mathbf{s},B\eta\mathbf{s}),\ \overline{(B\eta)}^2 = (\mathbf{s},(B\eta)^2\mathbf{s})$$

and analogous symbols for the tensor $\epsilon' B\eta$, then Eq. (43) can be rewritten

$$\left(\overline{\varepsilon'} + \frac{\overline{B\eta}}{n^2-\mu}\right)n^4 + \left[\overline{\varepsilon'^2} - \overline{\varepsilon'}\,\mathrm{Sp}\,\varepsilon' + \frac{\overline{\varepsilon' B\eta} + \overline{B\eta\varepsilon'} - \overline{\varepsilon'}\,\mathrm{Sp}\,B\eta - \overline{B\eta}\,\mathrm{Sp}\,\varepsilon'}{n^2-\mu} + \frac{\overline{(B\eta)}^2 - \overline{B\eta}\,\mathrm{Sp}\,B\eta}{(n^2-\mu)^2}\right]n^2 + \Delta\left(\overline{\varepsilon'} + \frac{B\eta}{n^2-\mu}\right) = 0. \tag{44}$$

In terms of n^2, this already is an equation whose order is higher than the second. Therefore, for a give s it determines, generally speaking, more than two values of |n| and correspondingly more than two plane waves. It will be shown below that the order of Eq. (44) depends on the crystal structure and on the direction of s.

We shall consider as examples, rhombic, tentragonal, and cubic crystals. In these crystals, the calculations are simplified because the principal axes of ϵ' and B coincide with the second or fourth order crystal axes, and consequently the directions of the principal axes are independent of ω, s, or n. In these cases, it is convenient to use the principal axes of ϵ' and B as Cartesian coordinate axes, as is done below.

Rhombic Crystals

In these crystals only one element of the tensor a does not vanish. To show this, we note that from among the three components of the vector $\mathbf{P}_0$ only one can differ from zero. In fact, in accordance with Eq. (8).

$$\mathbf{P}_0 = \int \Psi^{0*}\hat{\mathbf{P}}(0)\,\Psi_0 d\Omega. \tag{45}$$

Let us consider first crystals of the symmetry group D_2. Let $\hat{C}_x$, $\hat{C}_y$, and $\hat{C}_z$ be the operations of rotating all the electrons of the crystal and deforming the lattice through 180° around the second-order crystal axes, i.e., around the axes x, y, and z. The application of any one of these operators to any of the functions in the integrand in Eq. (45) is equivalent to changing the sign of some of the variables of integration and therefore does not change the value of the integral. Application of the well-known methods of group theory shows that only those components of the matrix element in Eq. (45) can be non-vanishing, for which the integrand is invariant for any transformation C. To determine how the integral transforms as a result of the rotations described above, let us consider how each of its three terms transforms.

The function Ψ^0, as the wave function of the ground state of the crystal, must be completely symmetrical, and consequently, invariant under all three rotations. The component of $\hat{\mathbf{P}}(0)$ on the rotation axis stays invariant, while the other components change signs. Since the point symmetry group of a rhombic crystal has only first order irreducible representations, Ψ_0 (and Ψ^0) are certainly eigenfunctions of all three operators $\hat{C}$:

$$\hat{C}_x\Psi_0 = C_x\Psi_0,\ \hat{C}_y\Psi_0 = C_y\Psi_0,\ \hat{C}_z\Psi_0 = C_z\Psi_0.$$

Since $\hat{C}_x^2 = \hat{C}_y^2 = \hat{C}_z^2 = 1$, the eigenvalues C_x, C_y, and C_z can have either one of two values, ± 1. Applying to Ψ_0, on both the left and right sides, the well-known operator equation $\hat{C}_x\hat{C}_y = \hat{C}_z$, we obtain $C_xC_y = C_z$. Hence, it follows that all three eigenvalues of $\hat{C}$ are equal to + 1, or else one is + 1 and the remaining two are − 1.

Taking into account the above transformation properties of the terms in the integrand of Eq. (45), it is easy to obtain the following results:

1. The case $C_x = C_y = C_z = 1$. All components of Eq. (45) vanish.
2. The case when one of the eigenvalues C (to be specific let this be C_x) is equal to 1, and the other two are equal to − 1; $C_x = 1$, $C_y = C_z = -1$. Then $P_{0x} \neq 0$, $P_{0y} = P_{0z} = 0$. Therefore, in accordance with Eqs. (14) and (41), the only non-vanishing components of the tensors a and B are a_{xx} and B_{xx}, which is what we needed to demonstrate. The same results are obtained also in rhombic crystals with symmetry groups C_{2v} and D_{2h}.

Thus in crystals with rhombic symmetry for any direction of the $\mathbf{E}$ vector, the right side of the vector equation (14) and P_1 will be directed along the x axis. This does not necessarily mean that there is a fundamental difference between this axis and the other principal axes of the crystal. This result was obtained because the discussion above was restricted to one exciton band whose wave functions are oriented along the x axis. But in the same crystal there exist other (not considered at this time) exciton bands, oriented preferentially along the y and z directions. The presence of several exciton bands can be taken into account, by introducing into Eq. (1) several ψ-like terms. The result is that in the right hand side of Eq. (5) several P_1-like vector terms appear, among which there are terms directed along the y and z axes. However, in rhombic crystals several of the exciton bands described above will not be degenerate in energy, and therefore, if the frequency of the light falls in the exciton absorption region of one of these bands, it will turn out to be outside the exciton absorption region of the other bands, and the existence of the latter can be taken into account simply by changing the constants ϵ' [see the remarks after Eq. (41)].

Let $b = B_{xx}$ denote the one non-vanishing component of the B tensor. When $\mathbf{s}$ is parallel to P_0, i.e., when $s_x = 1$, $s_y = s_z = 0$, the tensor $B\eta$ vanishes and therefore Eq. (44) assumes exactly the same form which it would have in ordinary crystal optics, when all polarizations can be simply characterized by the ϵ' tensors. In this case, no new answers are obtained; there exist two plane waves, polarized in the directions of the y and z axes with indices of refraction essentially equal to $\sqrt{\epsilon'_{yy}}$ and $\sqrt{\epsilon'_z}$.

However, for other directions of $\mathbf{s}$, for example, if $s_x = s_y = 0$ and $s_z = 1$, Eq. (44) becomes

$$(n^2 - \varepsilon'_{yy})\,[n^2 - \varepsilon'_{xx} - b/(n^2 - \mu)] = 0. \tag{46}$$

In this case the system of equations (22) has these three solutions

$$E_x = E_z = 0, \quad E_y \neq 0, \quad n = \sqrt{\varepsilon'_{yy}}; \qquad (47$$

$$E_y = E_z = 0, \quad E_x = \begin{cases} E_+ \quad n_+^2 = {}^1/_2 (\mu + \varepsilon'_{xx}) + \sqrt{{}^1/_4 (\mu - \varepsilon'_{xx})^2 + b}, & (48 \\ E_- \quad n_-^2 = {}^1/_2 (\mu + \varepsilon'_{xx}) - \sqrt{{}^1/_4 (\mu - \varepsilon'_{xx})^2 + b}. & (49 \end{cases}$$

Equation (47) is identical with the corresponding solution of ordinary crystal optics, but Eqs. (48) and (49) are new kinds of solutions. In order to satisfy the boundary condition $\mathbf{P}_1 = 0$ at the crystal surface (let this be the $z = 0$ surface) it is necessary to make up the following linear combination of Eqs. (48) and (49)

$$E_x (z, t) = E_+ (e^{i\omega n_+ z/c} - q e^{i\omega n_- z/c}) e^{-i\omega t}, \quad q = (n_-^2 - \mu) / (n_+^2 - \mu). \qquad (50$$

It is readily shown that, depending on whether ω deviates from the exciton photoabsorption band on the red or on the violet side, q becomes much less or much greater than unity. Thus in Eq. (50) the amplitude of one of the waves becomes vanishingly small compared with the amplitude of the other. The wave with the dominant amplitude goes over into the well known solution of ordinary crystal optics. However, in the vicinity of the exciton absorption band, both waves have amplitudes of the same order of magnitude and the solution is essentially different from the results of ordinary crystal optics. It should be noted, for example, that the solutions turn out to be superpositions of two waves with different indices of refraction, but the same polarization. As can be seen from Eqs. (48) and (49), none of the indices of refraction exhibit the customary hyperbolic discontinuity as functions of ω in the region of the exciton photoabsorption. Other theoretical formulae are also changed substantially. For example, the reflection coefficient for light on the surface of the crystal, for normal incidence along the z axis, is given by

$$R = \left| \frac{1 - q - n_+ + q n_-}{1 - q + n_+ - q n_-} \right|^2. \qquad (5$$

As one moves away from the exciton absorption region, when $q \ll 1$ or $q \gg 1$, Eq. (51) transforms into the formula of ordinary crystal optics.

Tetragonal Crystals

Let the x axis coincide with a fourth-order crystal axis, i.e., the lattice constants in the x and y directions are equal. An investigation of exciton states of such a crystal by the methods of group theory, which proceeds in a similar way to that used above for rhombic crystals, leads to the following results. The exciton bands can be subdivided into two types:

a) Non-Degenerate Exciton Bands. For these the vector $\mathbf{P}_0$ is directed along the x axis. If ω approaches the photoabsorption region of an exciton of this type, the tensor B has only one non-vanishing component, namely $B_{xx} = b$. Therefore, the same solutions of Eqs. (42) and (41) are obtained for such excitons as for rhombic crystals.

b) Doubly Degenerate Exciton Bands. One of them corresponds to the vector $\mathbf{P}_0$ directed along the y axis and the second to a vector $\mathbf{P}_0$ of the same magnitude but directed along the z axis. Let us denote these vectors by $\mathbf{P}_0^{(2)}$ and $\mathbf{P}_0^{(3)}$ respectively. To consider this case, it is necessary to introduce two ψ-like terms into the right side of Eq. (1). This leads to an equation similar to Eq. (14), in which however,

$$a_{xy} = \omega_0 V (P_{0x}^{(2)} P_{0y}^{(2)*} + P_{0x}^{(3)} P_{0y}^{(3)*}).$$

As a result

$$a = \omega_0 V \mid P_0^{(2)} \mid^2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad B = \begin{vmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{vmatrix}, \quad \varepsilon'_{yy} = \varepsilon'_{zz} \equiv \xi. \qquad (52$$

Consider a wave, for which s lies in the yz plane. Equation (44) changes into

$$(n^2-\varepsilon'_{xx})\,[n^2-\xi-b\,/\,(n^2-\mu)]=0. \tag{53}$$

Now the system (42) has three solutions:

$$E_y=E_z=0,\quad E_x\neq 0,\quad n=\sqrt{\varepsilon'_{xx}}; \tag{54}$$

$$\mathbf{E}=\begin{cases}\mathbf{E}_+ & n_+^2={}^1/_2(\mu+\xi)+\sqrt{{}^1/_4(\mu-\xi)^2+b}; \quad (55)\\ \mathbf{E}_- & n_-^2={}^1/_2(\mu+\xi)-\sqrt{{}^1/_4(\mu-\xi)^2+b}, \quad (56)\end{cases}$$

where $\mathbf{E}_+$ and $\mathbf{E}_-$ coincide in direction, must lie in the yz plane, and must be perpendicular to **s**. If the surface of the crystal is perpendicular to s, its equation is $\mathbf{s}\cdot\mathbf{r}=0$. In order to satisfy the boundary condition $\mathbf{P}_1=0$ on this surface, one must, in accordance with Eqs. (55) and (56), form the linear combination

$$\mathbf{E}(\mathbf{r},\ t)=\mathbf{E}_+\,[e^{i\omega n_+(\mathbf{s},\mathbf{r})/c}-qe^{i\omega n_-(\mathbf{s},\mathbf{r})/c}]\,e^{-i\omega t}. \tag{57}$$

The properties of this wave are completely analogous to the properties of the solutions of Eq. (50).

Let us now consider a wave, for which s is oriented along the x axis. Equation (44) then assumes the form

$$[n^2-\xi-b\,/\,(n^2-\mu)]^2=0. \tag{58}$$

The solutions of Eq. (42) then become

$$\mathbf{E}(x,\ t)=\mathbf{E}_+\,[e^{i\omega n_+x/c}-qe^{i\omega n_-x/c}]\,e^{-i\omega t}, \tag{59}$$

if the crystal surface is the plane $x=0$. Here $\mathbf{E}_+$ is an arbitrary vector lying in the yz plane, and n_+ and n_- have the same meaning they had in Eqs. (55) and (56).

We can also consider the case, for which s is an arbitrary direction in space. Then Eq. (44) is an ordinary fourth order equation in n^2. It can be solved if the components of the vector s are specified numerically.

Cubic Crystals

In these crystals the matrix element $\mathbf{P}_0$ is non-vanishing only for transitions in triply degenerate exciton bands. Three such bands correspond to vectors $\mathbf{P}_0$, equal in magnitude and oriented along the x, y, and z axes. Let us denote these vectors by $\mathbf{P}_0^{(1)}$, $\mathbf{P}_0^{(2)}$, and $\mathbf{P}_0^{(3)}$. Introducing three ψ-like terms into the right side of Eq. (1), corresponding to the three degenerate bands, we obtain Eq. (14), in which, however,

$$a_{xy}=\omega_0V\sum_{i=1}^{3}P_{0x}^{(i)}P_{0y}^{(i)*}.$$

Consequently, taking the directions of $\mathbf{P}_0^{(i)}$ into account, we obtain

$$a=\omega_0V\,|\,P_0^{(1)}\,|^2\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix},\quad B=b\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix},\quad \varepsilon'_{xx}=\varepsilon'_{yy}=\varepsilon'_{zz}\equiv\xi. \tag{60}$$

Thus the tensors B and ϵ' can be simply replaced by the numbers b and ξ. Then for any direction of s Eq. (44) assume the form of Eq. (58), and consequently, n_+ and n_- are given by Eqs. (55) and (56). The solution of (42) can be written

$$\mathbf{E}(\mathbf{r},\ t)=[\mathbf{E}_+e^{i\omega n_+(\mathbf{s},\mathbf{r})/c}+\mathbf{E}_-e^{i\omega n_-(\mathbf{s},\mathbf{r})/c}]\,e^{-i\omega t}, \tag{61}$$

where $\mathbf{E}_+$ and $\mathbf{E}_-$ must be normal to s but in other respects can have arbitrary orientations. Therefore, Eq. (61) contains four linearly independent waves. All of these represent new kinds of solutions. If the crystal surface is normal to s and coincides with the plane $\mathbf{s}\cdot\mathbf{r}=0$, then the boundary condition $\mathbf{P}_1=0$ yields a relation between $\mathbf{E}_+$ and $\mathbf{E}_-$ and Eq. (61) transforms into

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_+ \left[e^{i\omega n_+(\mathbf{s},\mathbf{r})/c} - q e^{i\omega n_-(\mathbf{s},\mathbf{r})/c}\right] e^{-i\omega t}. \tag{62}$$

Here $\mathbf{E}_+$ is any vector normal to $\mathbf{s}$. The reflection coefficient for light at the crystal surface is given by Eq. (51).

If the crystal surface is not perpendicular to $\mathbf{s}$, then in order to satisfy the boundary conditions, it is necessary to construct a linear combination of particular solutions with various values of $\mathbf{s}$. This remark is also applicable to the other crystal types which have been considered above.

Longitudinal Waves

All the solutions of Maxwell's equations described above were obtained under the assumption, that in Eq. (39) the denominator of the expression for $\mathbf{P}_1$ does not vanish. There exist, however, other exponential plane wave solutions of another type, in which n is determined just from the condition that this denominator should vanish, i.e., from the equation

$$\mathcal{E}(\omega n \mathbf{s} / c) = \hbar\omega. \tag{63}$$

In fact, in this case $\exp\{i\omega[n(\mathbf{s}\cdot\mathbf{r})/c - t]\}$ is a solution of the second of Eqs. (36), without the right side, i.e., with $\mathbf{E}_\perp = 0$. Then it follows from Eq. (38) that $\mathbf{D} = 0$, and the system of equations (35) will be satisfied, if one puts also $\mathbf{H} = 0$. It is still necessary to satisfy Eq. (36). As a result of solving the system of equations (35), Eq. (36) becomes

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i\omega[n(\mathbf{s},\mathbf{r})/c - t]};\quad \mathbf{E}_0 \parallel \mathbf{s};\quad \mathbf{P}_1 = -\frac{1}{4\pi}\varepsilon'\mathbf{E};\quad \mathbf{P}_2 \approx 0;\quad \mathbf{E}_\perp = \mathbf{D} = \mathbf{H} = 0. \tag{64}$$

For rhombic, tetragonal, and cubic crystals, it is easy to find linear combination solutions which satisfy the boundary condition $\mathbf{P}_1 = 0$. They have the following form:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \left[e^{i\omega n(\mathbf{s},\mathbf{r})/c} - e^{i\omega n(\tilde{\mathbf{s}},\mathbf{r})/c}\right] e^{-i\omega t}. \tag{65}$$

Here $\tilde{\mathbf{s}}$ is the specular reflection of $\mathbf{s}$ at the crystal surface. The waves represented by Eqs. (65) and (63) are also new solutions, not encountered in ordinary crystal optics. In the absence of exciton states in a medium such strictly-longitudinal waves cannot exist. It can be shown, that such waves cannot pass out of the crystal into the vacuum, which obviously make it difficult to detect them experimentally. Hence, such waves are a kind of natural polarization oscillations of the crystal. They represent macroscopic electric fields acting on the exciton. Their group velocity coincides with the group velocity of the exciton [this is readily obtained from Eq. (63)]. Since for such waves $\mathbf{D} = 0$, while $\mathbf{E} \neq 0$, their dielectric constant must be equal to zero. The simplest obvious case of such waves is the longitudinal optical oscillation of ions in the crystal. However, according to the theory presented above, similar waves should exist for all kinds of excitons.

In subsequent papers the necessary generalizations of crystal optics and methods of detecting the new waves experimentally will be investigated in detail.

The author expresses his thanks to L. D. Landau for very valuable discussions.

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Translated by J. J. Loferski

SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 4 APRIL, 1958

LETTERS TO THE EDITOR

CONVERSION OF L II AND L III ELECTRONS WITH EMISSION OF E1-QUANTUM IN E0 NUCLEAR TRANSITIONS

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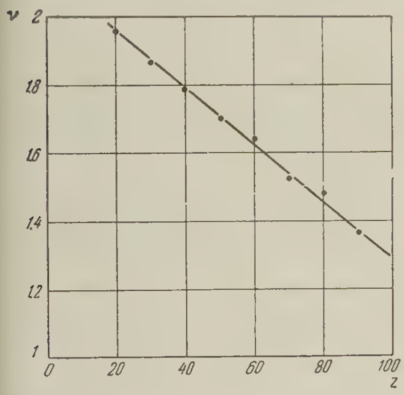
SHELL electron conversion in nuclear E0 transition is determined by the interaction in the region $r \leq R_0$ inside the nucleus. The probability of E0 conversion of the LII and LIII electrons is therefore substantially less than that of the LI electron (see Ref. 1).

$$W_{E0}(L\,\mathrm{II}) = {}^3\!/_2\,(Ze^2/2)^2\,p_f^2\,(\varepsilon_f+1)^{-2}\,W_{E0}(L\,\mathrm{I});$$
$$W_{E0}(L\,\mathrm{III}) \sim (p_fR_0)^2\,(Ze^2R_0/2)^2\,W_{E0}(L\,\mathrm{I}) \approx 10^{-8} \div 10^{-9}\,W_{E0}(L\,\mathrm{I}).$$

Here p_f, ϵ_f are the electron energy and momentum in the final state; relativistic units $\hbar = m = c = 1$ and $e^2 = 1/137$ are used throughout.

It would be interesting to consider conversion of LII and LIII electrons whereby the electron, emitting a dipole quantum (E1), changes to a state with momentum $j = 1/2$, $\ell = 0$ and then converts in the E0-nuclear transition. We give below the results of the calculations. We consider the emission of quanta with energies $k \leq Ze^2/2$; the calculations are carried out with the Coulomb functions of the electrons in the field of the nucleus in second-order perturbation theory. Screening is neglected. The finite dimensions of the nucleus are taken into account only in the calculation of the matrix element of the E0 transition.



After inserting the matrix elements of the E0 and E1 transitions, the integrals over the intermediate states for LII and LIII electrons reduce to integrals of the type

$$J = \int_0^\infty dx\,\frac{\Phi_\delta(x)\,x^m}{(1+x^2)^n\,(1+x^2+2\xi/\alpha)}, \quad x = \frac{p}{\alpha}, \quad m < n,$$

where

$$\alpha = Ze^2/2,\ \Phi_\delta(x) = x^{2\gamma-1}(1+x^2)^{3\delta/2}\exp(-4\varepsilon\ \tan^{-1} x/x), \quad \gamma = \sqrt{1-4\delta}, \quad \delta = \alpha^2, \quad \xi = k/\alpha.$$

To calculate this integral, Φ_δ is approximated in the interval $0.2 \leq x \leq 15$ by means of a power function $e^{-4+\lambda}x^\nu$. The dependence of ν on Z is given in the diagram. The value of λ is restricted to $0.9 \leq \lambda \leq 1.1$. Thus the permissible error does not exceed 20%. The calculated value of J is

$$J = \frac{\exp(-4+\lambda)}{2}\,I_n^{\nu m}\left(-\frac{2\xi}{\alpha}\right) = \frac{\exp(-4+\lambda)}{2\Gamma(n+1)}\,\Gamma\left(n+\frac{1-m-\nu}{2}\right)\Gamma\left(\frac{1+\nu+m}{2}\right){}_2F_1\left(1,\ n+\frac{1-m-\nu}{2};\quad n+1;\quad -\frac{2\xi}{\alpha}\right),$$

from which we obtain for the unknown probabilities

$$dW_{\gamma E0}(L\,\mathrm{II}) = 1.2\cdot 2^{4\gamma-2}\,[\Gamma(2\gamma+1)]^{-2}\,e^2W_{E0}(L\,\mathrm{I})\left\{\left[I_2^{\nu 0} + \frac{\gamma+1}{6}\,\xi^2\left(\frac{2\gamma+3}{3}\mathfrak{M} + 2I_3^{\nu 0} - (2\gamma+3)\,\mathfrak{N}\right)\right]^2.\right.$$
$$\left. + 2\left[\frac{(2\gamma+3)(2\gamma+2)}{45}\right]^2\xi^4\mathfrak{M}^2\right\}\delta(\Delta-\varepsilon_1-k-\varepsilon_f+1)\,\xi d\xi\,d\varepsilon_f, \quad \mathfrak{M} \approx I_5^{\nu 0} - I_5^{\nu 4} - 4I_5^{\nu 2}, \quad \mathfrak{N} \approx 0.6I_7^{\nu 5} + 2.5I_7^{\nu 6} - 5I_7^{\nu 2} - 3I_7^{\nu 0};$$

$$dW_{\gamma E0}(L\,\mathrm{III}) = 2.2\frac{2^{4\gamma-2}[\Gamma(\gamma_1+\gamma+1)]^2}{\Gamma(2\gamma_1+1)[\Gamma(2\gamma+2)]^3}W_{E0}(L\,\mathrm{I})\left\{\left[I_2^{\nu 0}+\frac{(\gamma+3)(\gamma+4)}{6}\xi^2\mathfrak{M}\right]^2\right.$$

$$\left.+\frac{9}{32}\left[\frac{(2\gamma+1)(\gamma+3)(\gamma+4)}{15}\right]^2\xi^4\left(\mathfrak{N}+\frac{5}{3}\mathfrak{M}\right)^2\right\}\delta(\Delta-\varepsilon_1-k-\varepsilon_f+1)\,\xi d\xi\, d\varepsilon_f,\quad \gamma_1=2\sqrt{1-\alpha^2},\quad \gamma=\sqrt{1-4\alpha^2},$$

where Δ is the nuclear transition energy and ϵ_1 is the binding energy of the L electron, equal to $(Ze^2)^2/8$.

If $\Delta \gg \alpha^2$ it is easy to estimate the lower limit of the total transition probability:

$$W_{\gamma E0}(L\,\mathrm{II}) \gtrsim W_{E0}(L\,\mathrm{I},\,\Delta)\frac{1}{137}1.2\frac{2^{4\gamma-2}}{[\Gamma(2\gamma+1)]^2}\frac{\alpha^2}{16}C,\quad W_{\gamma E0}(L\,\mathrm{III}) \gtrsim W_{E0}(L\,\mathrm{I},\,\Delta)\frac{1}{137}2.2\frac{2^{4\gamma-2}[\Gamma(\gamma_1+\gamma+1)]^2}{\Gamma(2\gamma_1+1)[\Gamma(2\gamma+2)]^3}\frac{\alpha^2}{16}C,$$

where the constant C is $\gtrsim 2$.

For the ratio of the two probabilities we obtain

$$W_{E0}(L\,\mathrm{II})/W_{\gamma E0}(L\,\mathrm{II}) \leqslant (1.37\cdot 10^3/C)\,2^{3-4\gamma}[\Gamma(2\gamma+1)]^2 p_f^2(\varepsilon_f+1)^{-2}.$$

if $\Delta \sim \alpha^2$ the ratio of the probabilities can be on the order of unity, owing to the smallness of p_f.

The ratio $W_{E0}(L\,\mathrm{III})/W_{\gamma E0}(L\,\mathrm{III})$ does not exceed 10^{-4} to 10^{-5} for all values of Δ.

Also possible for the LIII electron is a $\gamma - E0$ conversion with a transition first into the state $j = 1/2$, $\ell = 1$ and emission of a M1 and E2 quantum. In this case, however, the probability $dW_{\gamma E0}$ is proportional to $W_{E0}(L\,\mathrm{II})$ and is considerably less than the probability $dW_{\gamma E0}$ of a process with emission of an E1 quantum.

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SIGN OF THE SPACE CHARGE ON THE AXIS OF A POSITIVE COLUMN IN A LONGITUDINAL MAGNETIC FIELD

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THE magnitude of the space charge in a positive column, together with the magnitude of the transverse electric field, depends for a given current not only on the temperatures and masses of the particles and on the pressure, but also on the superimposed external longitudinal magnetic fields.[1] To calculate the deviation from the quasi-neutrality on the discharge axis one can use a scheme analogous to that proposed by Ecker.[2] However, since direct measurements of the excess space charge is difficult, it is advantageous to change over to transverse gradients.

The ratio of the transverse gradients near the axis in a magnetic field and without the field is given by

$$E_r^H/E_r = Z^H(D_e^H - D_p^H)(D_e b_p + D_p b_e)/Z(D_e - D_p)(D_e^H b_p^H + D_p^H b_e^H), \tag{1}$$

where D_e, D_p, b_e, and b_p are the coefficients of diffusion and mobility of the electrons and ions without a magnetic field (while the index H denotes the presence of a magnetic field), and Z is the ionization coefficient.

Under conditions usually existing in the positive column of a gas discharge, the transverse field decreases in the presence of a longitudinal magnetic field. The character of the decrease depends on the degree of non-isothermal nature of T_e/T_p and on the masses of the atoms of the gas in which the discharge takes place. For light gases the transverse field does not change its sign, while in a discharge in heavy gases there exists such a magnetic field, for which the sign of the transverse gradient reverses.* In the case of light elements relation (1) can be represented as

$$E_r^H / E_r = (Z^H / Z)(1 + b_p^2 H^2 / c^2), \tag{1a}$$

and for fields less than 1,000 gauss it reduces to

$$E_r^H / E_r = Z^H / Z. \tag{1b}$$

To compare (1b) with experimental results we employed measurements of the longitudinal gradient in neon with and without a magnetic field. The ratio of the transverse fields obtained from the longitudinal gradients with the aid of formula (1b) gave a result that was in satisfactory agreement with the measurements of Bicerton and Engel.[1]†

In the case of heavy elements Eq. (1) becomes

$$E_r^H / E_r = (Z^H / Z)\,[1 - (D_p b_e^2 - D_e b_p^2)\,H^2 / (D_e - D_p)\,c^2], \tag{1c}$$

and the magnetic field at which the transverse gradient vanishes is determined by

$$H_0^2 = c^2 (D_e - D_p) / (D_p b_e^2 - D_e b_p^2). \tag{2}$$

In the case of discharge in mercury vapor at a pressure 10^{-2} mm mercury and $T_e/T_p = 10^2$ the value of the magnetic field determined by (2) is approximately 2,000 gauss; at larger fields the transverse gradient should reverse its sign.‡ However, experiments in which such a phenomenon would be observed are unknown to this author, and it is therefore impossible to compare the calculated and experimental results.

In conclusion I feel it my duty to thank Professor Ia. P. Terletskii and A. A. Zaitsev for advice in various problems touched upon in this work.

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*When $T_e/T_p = 10^2$ and the free paths are gas-kinetic, the sign of the space charge cannot change for a gas of atomic weight less than 15.

†If the magnetic fields are large, the ratio (1a) tends to a constant value, as can be verified by using the equations from the Schottky theory.

‡The phenomenon can be studied at fields less than H_0. In this case the ratio of the gradients in a discharge in light gases should approach asymptotically a constant value, something that does not happen for heavy gases.

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CONTRIBUTION TO THE THEORY OF STRIPPING AT HIGH ENERGIES

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SERBER[1] determined the cross section of the stripping reaction under the assumption that the nuclear radius R is considerably greater than the deuteron radius R_d. This, however, is a poor assumption even for the heaviest nuclei, where the ratio $p = R/R_d$ reaches approximately 5. It is therefore desirable to determine the cross section of the stripping reaction without assuming $R \gg R_d$. This communication is devoted to this problem.

Let us consider, to be specific, a reaction that causes a nucleus to liberate a neutron and absorb a proton. This process can be described by a wave function $\Psi = \Omega_n \psi_0(\rho_d)\varphi_0(r)$, where $\varphi_0(r) = \sqrt{\alpha/2\pi} \times e^{-\alpha}/r$ is the wave function of the deuteron ground state $(\alpha = 1/2R_V)$, $\psi_0(\rho_d) = 1$ is the portion of the wave function describing the motion of the center of gravity of the deuteron in a plane perpendicular to the momentum of the incident deuteron, and Ω_n is a factor allowing for the absorption of neutron by the nucleus; this factor, in the case of an absolutely black nucleus, is

$$\Omega_n \equiv \Omega(\rho_n) = \begin{cases} 0 & \rho_n \leqslant R \\ 1 & \rho_n > R \end{cases}$$

(ρ_n is the projection of the neutron radius vector on the plane perpendicular to the momentum p_0 of the deuteron).

Expanding Ψ into a Fourier integral in the functions $e^{-i\mathbf{k}\mathbf{r}_n}$ ($\mathbf{r}_n$ is the radius vector of the neutron) we determine the amplitude of the probability $a_{\mathbf{k}}(\mathbf{r}_p)$ that the neutron has a wave vector $\mathbf{k}$ and the proton is located at the point $\mathbf{r}_p$:

$$a_{\mathbf{k}}(\mathbf{r}_p) = \int e^{-i\mathbf{k}\mathbf{r}_n} \Omega_n \varphi_0(r)\, d\mathbf{r}_n.$$

Integrating $|a_{\mathbf{k}}(\mathbf{r}_p)|^2$ with respect to $d\rho_p$ from $\rho_p = 0$ to $\rho_p = R$ we obtain the differential stripping cross section $d\sigma_n$ for which the wave vector of the neutron lies in the interval $d\mathbf{k}$.

$$d\sigma_n = \frac{d\mathbf{k}}{(2\pi)^3} \int\limits_{\rho_p < R} d\rho_p\, |a_{\mathbf{k}}(\rho_p)|^2 = \frac{d\mathbf{k}}{(2\pi)^3} \int d\rho_p \{1 - |\Omega_p|^2\}\, |a_{\mathbf{k}}(\rho_p)|^2,$$ (1

where $\Omega_p = \Omega(\rho_p)$.

The total stripping cross section is obviously

$$\sigma_n = \iint d\rho_p d\mathbf{r}_n \{1 - |\Omega_p|^2\}\, |\Omega_n|^2\, \varphi_0^2(r).$$ (2

expanding $\{1 - |\Omega(\rho)|^2\}$ into a Fourier integral

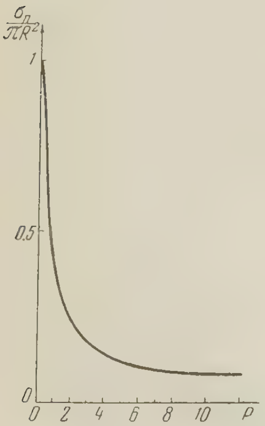
$$1 - |\Omega(\rho)|^2 = \int \chi(g)\, e^{i\mathbf{g}\rho}\, d\mathbf{g}, \quad \chi(g) = RJ_1(gR)/2\pi g,$$

where $J_1(x)$ is the Bessel function, we get

$$\sigma_n = \pi R^2 \left\{ 1 - 2\int_0^\infty \frac{p}{\xi} \tan^{-1} \frac{\xi}{p} \frac{J_1^2(\xi)}{\xi} d\xi \right\}.$$ (3

In the limit for large p this formula becomes the Serber formula

$$\sigma_n^{(s)} = \pi R R_d / 2.$$ (4



The diagram shows the variation of σ_n with p. In the case of lead $p = 4.2$ and Eq. (3) yields $\sigma_n = 3.2 \times 10^{-25}$ cm², while Serber's formula gives $\sigma_n^{(s)} = 2.$ $\times 10^{-25}$ cm². When $p = 1$, $\sigma_n = 5.8 \times 10^{-26}$ cm² and $\sigma_n^{(s)} = 6.9 \times 10^{-26}$ cm².

Formula (3) determines also the stripping cross section σ_p of the proton.

To obtain the energy distribution of the freed neutrons it is necessary to integrate (1) over the perpendicular components of the vector $\mathbf{k}$:

$$d\sigma_n(k_z) = \frac{dk_z}{2\pi} \int \frac{\partial \varkappa}{(2\pi)^2} \int d\rho_p \{1 - |\Omega_p|^2\} \left| \int d\rho_n e^{-i\varkappa\rho_n}\, \Omega_n \int dz e^{ik_z z} \varphi_0(r) \right|^2,$$

where κ is the projection of vector $\mathbf{k}$ on the plane perpendicular to p_0. Using the completeness of the functions $e^{i\kappa\rho_n}$ and expanding $|\Omega_n|^2$ and $|\Omega_p|^2$ into Fourier integrals, we obtain finally

$$d\sigma_n(k_z) = F(k_z)\, dk_z, \quad k_z = (E - E_0/2)/\hbar\sqrt{E_0/M},$$

$$F(k_z) = \frac{8pR^3}{\pi} \int_0^1 K_0^2 (p\sqrt{1 + k_z^2/\alpha^2}\,\zeta)(\sin^{-1}\zeta + \zeta\sqrt{1-\zeta^2})\,\zeta d\zeta, \tag{5}$$

where $K_0(x)$ is the modified Bessel function, E_0 is the energy of the incident deuteron, and M the neutron mass.

In the limiting case $p \gg 1$ this formula becomes the Serber formula

$$d\sigma_n^s(k_z) = (\pi/4)\,RR_d\alpha^2 dk_z/(\alpha^2 + k_z^2)^{3/2}. \tag{6}$$

Let us determine also the deuteron absorption cross section σ_a. Since[2]

$$\sigma_a + \sigma_n + \sigma_p = \sigma_t/2,$$

where

$$\sigma_t = 4\pi R^2 \left\{ 1 - \int_0^\infty \frac{p}{\xi} \tan^{-1} \frac{\xi}{p} \frac{J_1^2(\xi)}{\xi} d\xi \right\},$$

is the integral cross section for all the interactions between fast deuterons and nuclei, then

$$\sigma_a = 2\pi R^2 \int_0^\infty \frac{p}{\xi} \tan^{-1} \frac{\xi}{p} \frac{J_1^2(\xi)}{\xi} d\xi. \tag{7}$$

When $p \gg 1$ we get

$$\sigma_a = \pi R^2 - \pi RR_d/2. \tag{8}$$

It is possible to determine the influence of the Coulomb field and of the semi-transparency of the nucleus, as was done in Ref. 2. It is easy to see that the Coulomb field affects neither the total cross section nor the energy distribution of the particles. The semi-transparency of the nucleus decreases the stripping cross section. If the absorption is large, i.e., $|b|R \gg 1$ and if $p \gg 1$, then

$$\sigma_n = (\pi/2)\,RR_d\{1 - (1/2|b|^2R^2)\}. \tag{9}$$

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PARAMAGNETIC RESONANCE IN NEODYMIUM NITRATE

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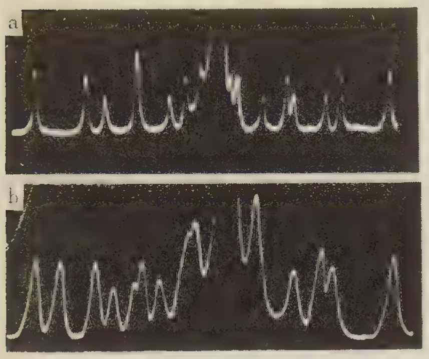
THE paramagnetic resonance spectra of rare-earth ions have been intensely investigated in recent years by Bleany and his colleagues. The measurements were made primarily on ethyl-sulphates $M(C_2H_5SO_4)_3 \cdot 9H_2O$, where M is a rare-earth ion. A study of the paramagnetic resonance of rare-earth ions in other compounds is also of great interest. At the suggestion of S. A. Al'tshuler and B. M. Kozyrev, we began an investigation of the nitrates of rare-earth elements, $M(NO_3)_3 \cdot 6H_2O$.

The measurements were made at a wavelength of 3.2 cm at liquid-hydrogen temperatures using a balanced superheterodyne radio-spectroscope of the type described by Manenkov and Prokhorov.[1] We investigated the paramagnetic resonance spectrum of neodymium nitrate, diluted with lanthanum in the ratio Nd : La = 1 : 200.

The spectrum consists of one intense line and 16 weak lines of the hyperfine structure (see diagram). This is in full agreement with the measurements of the paramagnetic resonance spectrum of neodymium in ethyl sulphate.[2,3]

Natural neodymium has isotopes with atomic weights 142, 143, 144, 145, 146, 148, and 150. The hyperfine structure is due to the odd isotopes 143 and 145. The remaining isotopes have a nuclear spin I = 0 and yield one strong line. The Nd^{143} content is 12.2%, and that of Nd^{145} is 8.3%. Accordingly, the hyperfine structure is broken up into two groups, each consisting of eight lines. This corresponds to a nuclear spin I = $^7/_2$ for each isotope. The line intensity in one group is approximately 1.5 times greater than that in the other group. The stronger lines were therefore attributed to Nd^{143} and the weaker ones to Nd^{145}.



Oscillograms of paramagnetic-resonance spectrum in neodymium nitrate: (a) x axis parallel to the constant magnetic field; hyperfine splitting lines almost equidistant; (b) y axis parallel to the constant magnetic field; strong second-order shifts are visible.

The line width varied linearly from 13 to 18° K, and broadened so much at 20° K to become unobservable. It must be noted that in ethyl sulphate practically no line broadening due to spin-lattice relaxation was observed even at 20° K.

The line width in the nitrate was inversely proportional to the g-factor even at 13° K. This indicates that the spin-lattice relaxation makes the major contribution to the line width.[4]

The spectrum is described by the following spin Hamiltonian:

$$\mathcal{H} = \beta\,(g_xH_xS_x + g_yH_yS_y + g_zH_zS_z) + A_xS_xI_x + A_yS_yI_y + A_zS_zI_z + P_xI_x^2 + P_yI_y^2 + P_zI_z^2,$$

where S is the effective spin, equal to ½; g_x, g_y, g_z are the principal values of the g-factor; H_x, H_y, H_z are the components of the constant magnetic field, A_x, A_y, A_z and P_x, P_y, P_z are the constants of the hyperfine and quadrupole interaction, I is the nuclear spin, and β is the Bohr magneton.

To find the x, y, and z axes we worked out together with A. M. Prokhorov and simple procedure, consisting essentially of first setting the crystal, mounted in an arbitrary manner, to the extremal value of the g-factor, cutting it in the plane of the axis of rotation and the magnetic field plane, and mounting it in this plane (provided the extremal g-factor is close to the principal value).

The principal values of the g-factors are

$$g_x = 3.88 \pm 0.01,\ g_y = 1.72 \pm 0.01,\ g_z = 0.74 \pm 0.01.$$

The following values were obtained for the hyperfine splitting constants (in units of 10^{-4} cm^{-1}):

$$\text{isotope 143: } A_x = 432 \pm 2,\ A_y = 193 \pm 2,\ A_z = 82 \pm 10,$$
$$\text{isotope 145: } A_x = 270 \pm 2,\ A_y = 119 \pm 2,\ A_z = 51 \pm 10.$$

It is possible to estimate the upper limit of the quadrupole coupling constants $|P_x - P_z| < 50 \times 10^{-4}$ cm^{-} and $|P_y - P_z| < 50 \times 10^{-4}$ cm^{-1} for both isotopes.

It is easy to see that the hyperfine splitting constants obey the following approximate equality:

$$A_x : A_y : A_z \approx g_x : g_y : g_z.$$

The ratios

$$A_x^{143} / A_x^{145} = 1.60 \pm 0.02,\ A_y^{143} / A_y^{145} = 1.62 \pm 0.04$$

are numerically equal to the ratios of the magnetic moments of the Nd^{143} and Nd^{145} nuclei and are in good agreement with results obtained by Bleaney et al. within the limits of experimental accuracy.[2,3]

In conclusion the author expresses his gratitude to E. L. Andronikashvili, A. M. Prokhorov, G. R. Khutsishvili, G. M. Mirianashvili, and A. A. Manenkov for valuable advice, counsel, and constant interest in this work, and also to D. S. Tsakadze, M. Koloch, and G. A. Tsinadze for taking part in the measurements.

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RADIATION CORRECTIONS TO PARTICLE SCATTERING IN EXTERNAL FIELD AND TO COMPTON EFFECT IN SCALAR QUANTUM ELECTRODYNAMICS

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THE expression for the single-photon mass operator of a scalar particle, obtained by one of the authors,[1] was used to calculate the radiation corrections to the scattering of a scalar particle in an external electromagnetic field and to the Compton effect. The following expressions were obtained thereby for the differential cross sections.

1. The differential scattering cross section (in the first Born approximation) has the following form

$$d\sigma / d\Omega = (d\sigma / d\Omega)_0 + (d\sigma / d\Omega)_{\Delta M} + (d\sigma / d\Omega)_{A'}, \tag{1}$$

where $(d\sigma/d\Omega)_0$ is the differential scattering cross section without allowance for radiation corrections, and the indices ΔM and A' distinguish the radiation corrections from the mass operator and from the polarization of vacuum, respectively.[2,3] For the first correction in (1) we have the following formula

$$(d\sigma / d\Omega)_{\Delta M} = -(2\alpha / \pi)(d\sigma / d\Omega)_0 \left[2y \coth 2y\,(h(2y) - h(y)) - y \tanh y + \ln \lambda\,(1 - 2y \coth 2y)\right],$$

$$h(y) = y^{-1} \int_0^y \varphi \coth \varphi\, d\varphi; \quad \sinh^2 y = (p_1 - p_2)^2 / 4m^2, \tag{2}$$

where λ is the photon mass in units of m; p_1 and p_2 are four-dimensional particle momenta before and after scattering, and $\alpha = e^2/4$. (We use a system of units in which $\hbar = c = 1$.)

For the second correction we have

$$(d\sigma / d\Omega)_{A'_0} = \frac{\alpha}{\pi}\left(\frac{d\sigma}{d\Omega}\right)_0 \left[\frac{4m^2 + (p_1 - p_2)^2}{3(p_1 - p_2)^2}(1 - y \coth y) + \frac{1}{9}\right] \tag{3}$$

if the vacuum polarization is due to particles with zero spin, and

$$(d\sigma / d\Omega)_{A'_{1/2}} = -\frac{2\alpha}{\pi}\left(\frac{d\sigma}{d\Omega}\right)_0 \left[\frac{4m_{1/2}^2 - 2(p_1 - p_2)^2}{3(p_1 - p_2)^2}(1 - y_{1/2} \coth y_{1/2}) + \frac{1}{9}\right], \quad \sinh^2 y_{1/2} = (p_1 - p_2)^2 / 4m_{1/2}^2, \tag{4}$$

if the vacuum polarization is due to particles with spin ½.

When λ approaches zero, formula (2) diverges logarithmically. This divergence is offset by an analogous divergence in the inelastic-scattering differential cross section of the particle with emission of a single soft photon.

$$(d\sigma / d\Omega)_{\text{inel}} = -\frac{2\alpha}{\pi}(d\sigma / d\Omega)_0 \left\{(1 - 2y \coth 2y)\left[\ln\left(\frac{K_{\max}}{\lambda}\right) - \frac{1}{2}\right] + 4y \coth 2y\,(h(2y) - 1)\right\}, \tag{5}$$

where $K_{\max}$ is the maximum momentum of the emitted photon in units of m ($K_{\max} \ll 1$).

2. The differential cross section of the Compton effect on a scalar particle has the following form

$$d\sigma / d\Omega = (\alpha / m)^2 (q_{20} / q_{10})^2 U, \qquad (6$$

where $d\Omega$ is the element of solid angle in the direction of the scattered photon and q_{10} and q_{20} are the energies of the incident and scattered photons in the laboratory system of coordinates (q_1 and q_2 are the photon four-momenta)

$$U = U^{(0)} + U^{(1)}(\varkappa, \tau) + U^{(1)}(\tau, \varkappa), \ U^{(0)} = (\varepsilon_1\varepsilon_2)^2,$$

$$\begin{aligned} U^{(1)}(\varkappa, \tau) = & -\frac{\alpha}{\pi}(\varepsilon_1\varepsilon_2)^2 \operatorname{Re}\left\{\frac{3}{4} + (1 - 2y\coth 2y)\ln\lambda + 2y\coth 2y\,(h(2y) - h(y)) - \frac{y^2}{\sigma} - \frac{\varkappa}{\varkappa - 1}\ln\varkappa \right. \\ & \left. + \frac{\sigma - 2}{\sigma - \varkappa\tau}\left[2y\coth y\varkappa\,(h(y) - \ln\varkappa) + (\varkappa - 2)\left(y^2 + \frac{\varkappa}{2}C_0(\varkappa)\right)\right]\right\} - \frac{\alpha}{\pi}\frac{(\varepsilon_1\varepsilon_2)(\varepsilon_1 q_2)(\varepsilon_2 q_1)}{m^2}\operatorname{Re}\left\{\frac{1}{2\sigma} + \frac{2y^2}{\sigma^2} + \frac{2}{\varkappa - 1}\ln\varkappa \right. \qquad (7 \\ & - \frac{2(\sigma - 2)}{(\sigma - \varkappa\tau)^2}\left[2y\coth y\varkappa\,(h(y) - \ln\varkappa) + (\varkappa - 2)\left(y^2 + \frac{\varkappa}{2}C_0(\varkappa)\right)\right] \left. - \frac{2(\sigma - 2)}{\sigma - \varkappa\tau}\left[\frac{4\tau}{\sigma(\sigma - 4)}y\coth y\,(h(y) - \ln\varkappa) - \frac{\tau}{\sigma}\ln\varkappa\right]\right\}, \end{aligned}$$

$$m^2\varkappa = 2p_1q_1; \ m^2\tau = -2p_1q_2; \ \sigma = \varkappa + \tau, \ \sinh^2 y = -\sigma/4; \quad h(y) = y^{-1}\int_0^y \varphi\coth\varphi\, d\varphi; \ C_0(\varkappa) = -2\varkappa^{-1}\int_{1-\varkappa}^{1}\ln(1 - u)\frac{du}{u}.$$

The photon polarizations are chosen to satisfy the relations $\epsilon_1\epsilon_1 = \epsilon_2\epsilon_2 = 1$, and $\epsilon_1 p_1 = \epsilon_2 p_1 = 0$.*

Formula (7) contains terms with $\ln\lambda$. These terms cancel each other out in the total cross section of the ordinary Compton effect and in the Compton effect accompanied by emission of one soft photon (the so-called double Compton effect).† Let us note that in the nonrelativistic approximation the formul for the total cross section of the above processes is the same as obtained in Ref. 5.

In conclusion, the authors are grateful to Professor A. I. Akhiezer for supervising the work and to Г. I. Fomin for help in evaluating the integrals.

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*The properties of the functions h(y) and $C_0(\kappa)$ are treated in detail by Brown and Feynman.[4]

† The connection between the cross sections of the double and ordinary Compton effects is given by Eq. (7).

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MAGNETIC RESONANCE OF NUCLEI OF PARAMAGNETIC ATOMS

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WE have carried out a detailed theoretical investigation of the possibility of observing magnetic resonance in ionic crystals of nuclei of paramagnetic atoms of the iron group and of the rare earth group. The effect under consideration is intermediate between the well known phenomena of paramagnetic electron resonance and the usual nuclear resonance of nuclei of diamagnetic atoms with respect to the inten sity of the absorption lines and to the frequency of the rf field at which the resonance should be observed. As is well known, the intensity of the resonance line is proportional to the square of the frequency of the rf field and to the probability of magnetic dipole transitions between the energy levels. At the ordinarily employed values of the intensity of the external magnetic field ($\sim 10^3$ oersted) the nuclear resonance frequency of paramagnetic atoms ($10^8 - 10^9$ cps) will be 10 to 100 times greater than the resonance frequency in the case of diamagnetic atoms ($10^6 - 10^7$ cps). Further, owing to the influence of the magnetic moment of the electron shell of the ion, the probabilities of nuclear magnetic dipole transitions are also considerably increased — approximately by a factor 10^4 for ions with an odd number of electrons and by a factor of several fold for ions with an even number of electrons. From this it may be seen that nuclear resonance absorption in the case of paramagnetic atoms will be so strong that it should be easily observable with already existing apparatus. But the possibility of observing resonance absorption depends also on the line width: the line must be sufficiently narrow. The line width of magnetic resonance is determined by dipole-dipole and spin-lattice interactions of the magnetic particles. Magnetic interactions between particles can make a large contribution to the line width in undiluted crystals, but their role rapidly becomes less important as the crystal is magnetically diluted. In dilute crystals the line width will be determined only by the spin-lattice interactions of the nuclei.

We have calculated the probabilities of nuclear relaxation transitions for the Cr^{3+} ion (odd number of electrons) and for the V^{3+} ion (even number of electrons). From the calculations it follows that in chrome alums at liquid air temperatures the line width of nuclear resonance does not exceed a few oersted and the absorption may be observed by the usual method of modulating a constant magnetic field. In salts of the vanadium ion the spin-lattice nuclear relaxation time will be short and it will be possible to observe nuclear resonance only at low temperatures. A situation favorable for experiments on nuclear resonance exists in crystals of Tutton salts of the ion Ni^{2+} enriched in the isotope of mass 61 (nuclear spin $^5/_2$).

We have also calculated the probabilities of nuclear relaxation transitions in crystals of ethyl sulphates of rare earth elements. We have investigated electron levels two-fold degenerate in the crystalline field and possessing electron magnetism. The probabilities calculated for the nuclear spins were compared with the probabilities of relaxation transitions between electron levels. For most ions with an odd number of electrons the probabilities of nuclear transitions are approximately 100 times smaller than the probabilities of electronic transitions, while for ions with an even number of electrons they are smaller by a factor 10^5. One may expect that it should be possible to observe nuclear resonance in the case of ions with an even number of electrons at liquid air temperatures, and for ions with an odd number of electrons at liquid hydrogen temperatures.

By using the method of magnetic resonance of nuclei of paramagnetic ions (by means of measuring the resonance transition frequencies) one can determine quite accurately the magnetic and quadrupole coupling constants of the nuclei of paramagnetic atoms. But the most interesting application of nuclear resonance in the case of paramagnetic atoms will evidently be investigations of spin-lattice interaction in crystals, particularly in those cases when direct measurements of the electronic relaxation time by the method of parallel fields or by the method of saturation are impossible due to very short relaxation times (in salts of the rare earth elements and in some salts of ions of the iron group). It should be possible to determine the nuclear spin relaxation time from the line width of nuclear resonance. On the other hand, the ratio of the probabilities of electronic and nuclear relaxation transitions may be calculated with suffient accuracy. Thus, for the Cr^{3+} and the V^{3+} ions, it has the form:

$$A(m,\ m-1)/A(M-1,\ M) = {}^1/_4\,(S+M)(S-M+1)(I+m)(I-m+1)(A/D)^2,$$

where S and I are the electron and the nuclear spin of the ion, A is the coupling constant between the nuclear and the electron spin of the ion, and D is the energy interval between neighboring energy levels of the electron spin. We have also calculated similar ratios for the neodymium and praseodymium ions.

The results of these calculations will be published in greater detail in the journal "The Physics of Metals and Metal Research" and in the Scientific Notes of the Kazan' University.

The author expresses his deep gratitude to S. A. Al'tshuler for suggesting the problem and for his interest in it in the course of its solution.

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ELASTIC SCATTERING OF $\pi^{\pm}$ MESONS BY HELIUM NUCLEI AT 300 MEV

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In an earlier note[1] we presented some preliminary results of an investigation of the interaction of negative π mesons with helium nuclei. We called attention to the considerable decrease in elastic scattering cross section which takes place at small angles (5 — 15°). This variation in the differential cross section was linked to a possible interference between Coulomb and nuclear scattering.

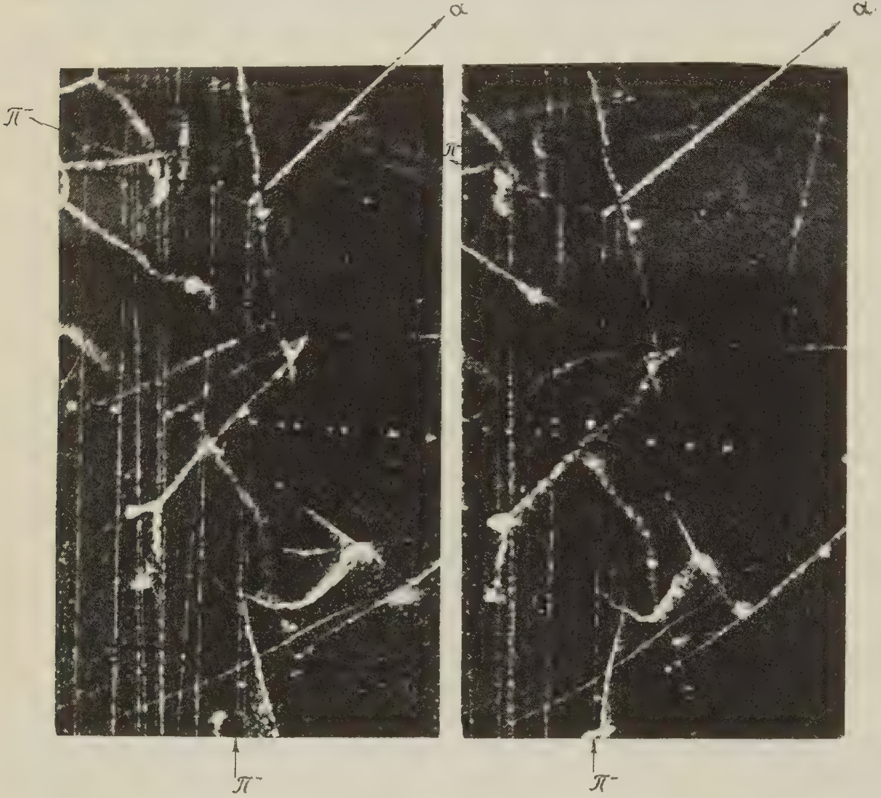

FIG. 1. Photograph of a 300-Mev π^- meson elastically scattered by a helium nucleus at an angle of 9.8°.

The present note describes further experiments on the elastic scattering of π^- and π^+ mesons against helium nuclei near 300 Mev carried out with a considerably sharpened angular distribution, and verifies the conclusion made about the role of Coulomb interference.

As in the previous experiment, the elastic scattering was studied by means of a diffusion cloud chamber[2] filled with helium at a pressure of 15 atmospheres. The geometry of the outcoming beam of negative π mesons and the experimental set-up were described in Refs. 1 and 2. The positive π mesons were generated in a polyethylene target 20 cm thick, which was placed in the path of a 660-Mev proton beam emerging from the synchro-cyclotron. An analyzing magnet selects a monochromatic beam of π^+ mesons which are formed as a result of the reaction $p + p \rightarrow \pi^+ + d$, and they emerge at an angle of 9° to the proton beam. After passing through the analyzing magnet, the beam of π mesons is directed through a collimating slit in the 4 m concrete shielding into the experimental space which contains the diffusion cloud chamber. In order to remove from the meson beam protons which have the same momentum, a 6 cm thick carbon filter was placed in front of the chamber. In the experiment with π^- mesons having an energy of 330 ± 6 Mev, we obtained altogether 24,000 photographs, while π^+ mesons of 273 ± 7 Mev yielded 11,000 photographs.

The elastic scattering of π mesons by helium nuclei were identified by the kinematics of the diverging particles: coplanarity, appropriate angular correlation, and range of the α-particles. For small-angle scattering, coplanarity and angular correspondence can only be crudely determined. In that case, a more reliable criterion is provided by the correspondence between the scattering angle of the mesons and the range of the α-particles. A small-angle region determined by the maximal angle of $\pi - \mu$ decay (5.1° at 300 Mev and 5.9° at 273 Mev) was excluded from consideration. A typical photograph of an elastic scattering event is shown in Fig. 1.

After twice scanning the pictures and analyzing the photographs, we detected 99 cases of elastic scattering of π^- mesons and 76 cases of π^+ mesons. The total absolute cross sections for elastic scattering

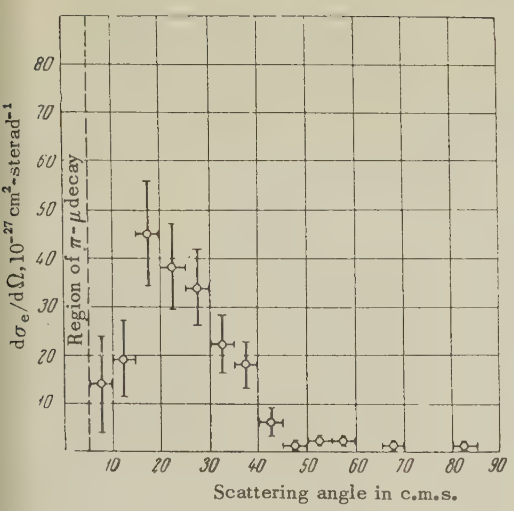

FIG. 2. Angular distribution of 330-Mev π^- mesons elastically scattered by helium nuclei.

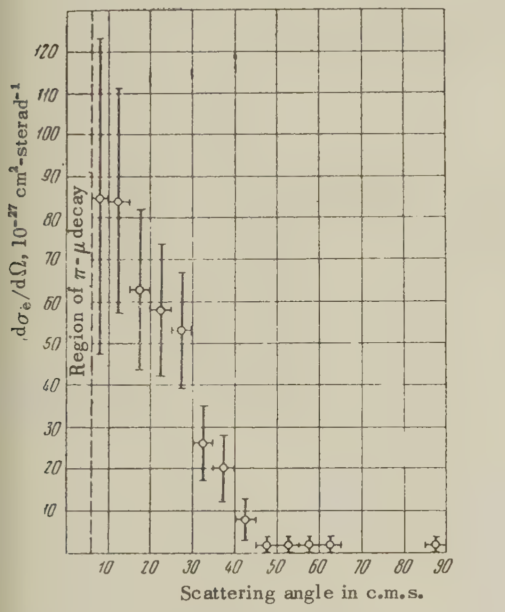

FIG. 3. Angular distribution of 273-Mev π^+ mesons elastically scattered by helium nuclei.

are respectively $(45 \pm 5) \times 10^{-27}$ cm^2 and $(72 \pm 11) \times 10^{-27}$ cm^2. The angular distributions in the center-of-mass system are presented in Figs. 2 and 3. The results show quite definitely the presence of an interference effect between Coulomb and nuclear scattering in the small-angle region. Essentially, the sign of the nuclear scattering amplitude is positive in the energy region under consideration, i.e., the interaction between π mesons and nuclei is repulsive. A calculation of the energy dependence of particles interacting with an average nuclear potential has been carried out using the optical model of the nucleus, and utilizing results on the scattering of π mesons against free nucleons;[3,4] these indicate a change in the sign of the nuclear amplitude in the region of the first resonance maximum, and are thus in agreement with the obtained experimental results.

The authors would like to express their gratitude to L. I. Landus for his interest in this work and for helpful discussion.

APPENDIX

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Recently there has been renewed discussion on the possible existence of a non-zero spin for the π meson. This discussion is based on investigations of the angular distribution in $\pi-\mu$ decay.[5,6] In the Laboratory for Nuclear Problems we have tried to settle this question by studying the experimentally-obtained azimuthal distribution of π^+ mesons elastically scattered by nuclei, described in the above communication. Out of 76 cases of elastic scattering, 47 mesons were found scattered to the right and 29 to the left. An analysis of experimental conditions indicated that the observed asymmetry could not be due to systematic error. The probability for such an accidental deviation from isotropic distribution is less than 0.05. Since even such an improbable fluctuation may have occurred during an individual measurement, we are presently carrying out further experiments along this line. In view of the present discussion, however, we have deemed it worth while to announce the results of our preliminary investigations.

No azimuthal anisotropy in the angular distribution was observed in the elastic scattering of negative π mesons.

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NUCLEON ENERGY LEVELS IN SPHEROIDAL NUCLEI

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THE problem of energy levels in a nonspherical nucleus has been treated fully only for an oscillator potential.[1] The case of an ellipsoidal well was treated by perturbation theory with expansion in terms of the deformation parameter (see for example, Ref. 2). Because of poor convergence this treatment cannot be used both for highly excited states and for large deformations. The present note concerns a method by which the spectrum of highly excited states in an ellipsoidal well can be found for arbitrary deformations, except for vanishingly small ones.

For the case of an infinitely deep prolate spheroidal well the problem consists of solving the equation

$$(\hbar^2/2M)\,\Delta\Psi + E\Psi = 0$$

inside the spheroid with zero boundary conditions. We introduce the spheroid coordinates u, θ, and φ:

$$x = f\sinh u \sin\theta\cos\varphi,\quad y = f\sinh u\sin\theta\sin\varphi,\quad z = f\cosh u\cos\theta,$$

where $0 \leq u \leq u_0$, $0 \leq \theta \leq \pi$, and $2f$ is the distance between the foci of the spheroid.

Putting $\Psi = R(u)\,Y(\theta)\exp(\pm im\varphi)$ we obtain

$$d^2R/du^2 + \coth u\,dR/du + (\gamma^2\cosh^2 u - \Lambda - m^2/\sinh^2 u)R = 0,$$

$$d^2Y/d\theta^2 + \cot\theta\,dY/d\theta + (\Lambda - \gamma^2\cos^2\theta - m^2/\sin^2\theta)\,Y = 0,\quad \gamma^2 = 2f^2ME/\hbar^2,$$

where Λ is the separation constant. Let γ be large. Utilizing the boundedness of Y in the range $0 \leq \theta \leq \pi$ we can show that Λ can be given in the form[3,4]

$$\Lambda = q\gamma + m^2 - \frac{1}{8}(q^2+5) + \sum_{i=1}^{\infty}\lambda_i(l, m)/\gamma^i,\quad q = 2(l - |m|) + 1,$$

where $\ell - |m|$ is the number of nodes of the function Y in the interval $0 - \pi$ of θ. The functions $\lambda_i(\ell, m)$ are given by Meixner[3] and by Sips.[4]

We find the function $R(u)$ by a generalization of the method of Miller and Good[5] and obtain

$$R(u) = [S(u)/\sinh u(dS/du)]^{1/2} J_m(\gamma S(u)), \tag{1}$$

where $J_m(z)$ is the Bessel function, and

$$S(u) = \sum_{k=0}^{\infty}\gamma^{-k}S_k(u),\ S_0(u) = \sinh u,\ S_1(u) = -\frac{q}{2}\cot^{-1}\sinh u,\ S_2(u) = -\frac{q^2+3}{16}\frac{\sinh u}{\cosh^2 u},$$

$$S_3(u) = -\frac{q}{128}(3q^2+41)\frac{\sinh u}{\cosh^2 u} + \frac{q}{64}(q^2+19)\frac{\sinh^3 u}{\cosh^4 u} + \frac{q}{4}\left(m^2 - \frac{1}{4}\right)\left(1 - \frac{\cot^{-1}\sinh u}{\sinh u}\right)\frac{1}{\sinh u}$$

etc.

Since $R(u_0) = 0$, the quantities $\gamma_{n\ell m}$ and with it the energies $E_{n\ell m}$ are obtained from the algebraic equation

$$\gamma S(u_0) = j_{mn}, \tag{2}$$

where j_{mn} is the n-th root of the function $J_m(z)$. Numerical calculations for a prolate spheroid with a ratio of the axes of 1.2 : 1 show that the use of four terms of the expansion of $S(u)$ in (2) allows the determination of all levels with an error ≤ 1%. The results for the lower levels agree with Moszkowski' results.[2]

We now take into account the finite depth of the potential well and the spin-orbit force. The operator for the latter[6] has in spheroidal coordinates the form

$$\hat{H}_{s0} = -\frac{\lambda\hbar^2}{2iM^2c^2}\,\frac{1}{f^2(\cosh^2 u - \cos^2\theta)}\,\frac{\partial V}{\partial u}\left[-\hat{\sigma}_1\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\sigma}_2\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\sigma}_3\coth u\frac{\partial}{\partial\varphi}\right], \quad (3)$$

where λ is the force constant, $\hat{\sigma}_i$ the Pauli spin matrices, and V the potential, which equals zero for $u < u_0$ and V_0 for $u > u_0$. From the symmetry of the problem it follows that only the third term of (3) contributes to the energy. To obtain the energy spectrum one has merely to solve the transcendental equation for γ, which follows from the matching conditions of the wave function at $u = u_0$:

$$\left[\gamma\frac{J_{m+1}(\gamma S)}{J_m(\gamma S)}\,\frac{dS}{du} - \frac{m + 1/2}{S}\,\frac{dS}{du} + \frac{1}{2}\left(\frac{dS}{du}\right)^{-1}\frac{d^2S}{du^2} - g\chi - \frac{1}{2}\,\frac{1}{\chi}\,\frac{d\chi}{du}\right]_{u=u_0} = -A,$$

where

$$g = f\sqrt{\frac{2M}{\hbar^2}(V_0 - E)},$$

$$\chi(u) = \sinh u + \frac{1}{g}\,\frac{p}{\sinh u} - \frac{p^2 - m^2 + 1}{2g^2}\left(\frac{1}{2\sinh u} + \frac{1}{\sinh^3 u}\right) - \frac{p}{2g^3}\left(\frac{p^2 - m^2 + 1}{8\sinh u} - \frac{p^2 - m^2 + 7}{2\sinh^3 u} - \frac{p^2 - m^2 + 5}{\sinh^5 u}\right) + \ldots,$$

$$A = \pm\lambda m V_0 \coth u_0/(Mc^2),$$

$p = \ell + 1$ for even $\ell - |m|$ and $p = \ell$ for odd $\ell - |m|$, and the two signs of A correspond to the two possible orientations of the nucleon spin with respect to the z axis.

The case of the oblate spheroid can be treated in an analogous manner.

Numerical computations of the energy levels of nucleons in ellipsoidal wells are being carried out at the present time for different deformations.

The author thanks M. V. Kazarnovskii and A. S. Davydov for useful suggestions and discussion of the results.

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DIFFRACTION SCATTERING OF HIGH-ENERGY PROTONS BY PROTONS

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FOWLER et al.[1] attempted a theoretical analysis of the diffraction p − p scattering, starting, however, with the usual spherical nucleon model with distinct boundaries and a definite transparency. Of greater interest is Belenkii's analysis[2] of the diffraction scattering of high-energy pions by nucleons, based on a general theory involving no specific nucleon model. We considered similarly the collisions of high-energy protons and analyzed the known experimental data, making the following assumptions: (1) the spin dependence of the nuclear forces can be neglected at high energies, (2) the imaginary part of the scattering amplitude is much greater than the real part (since it is known from the experimental data of Ref. 1 that the elastic scattering cross section is on the same order as the inelastic one at high energies and that both are on the order of the geometric cross section).

These assumptions simplify considerably the phase analysis, since (see Ref. 2) it follows from

$$|f(x)| = \frac{\lambdabar}{2}\sum_{l=0}^{\infty}(2l+1)(1-\beta_l)P_l(x). \qquad (1)$$

[where $f(x)$ is the scattering amplitude, $\lambdabar$ the wavelength of the incident particle, $x = \cos\vartheta$, ϑ the scattering angle, $P_l(x)$ the Legendre polynomials, $\beta_l = \exp(2i\eta_l)$, and η_l is the scattering phase] that

$$\beta_l = 1 - \lambdabar^{-1}\int_{-1}^{+1}|f(x)|P_l(x)\,dx.$$

By expressing the measured angular distribution in terms of an analytic function it is possible to determine β_l, since the differential cross section of the elastic scattering is $d\sigma/d\omega = |f(x)|^2$. The present-day experimental accuracy does not permit unique determination of the phase, but makes possible an approximate phase analysis, the results of which are given in the table.

We obtained formulas similar to (5) and (6) of Belen'kii's paper. Additional simple transformations yield

$$d\sigma(\vartheta)/d\omega = |f(\vartheta)|^2 = (\sigma_t/4\pi\lambdabar)^2\exp\{-R^2\vartheta^2/2\lambdabar^2\}, \qquad (2$$

where $R^2 = \sigma_t^2/8\pi\sigma_e$, where σ_e is the elastic scattering cross section and σ_t is the total cross section of particle interaction.

In the analysis of the experimental data obtained by Fowler et al. for proton energies of 1.5 and 2.75 Bev, we calculated the $d\sigma(\vartheta)/d\omega$ curves from values of σ_t known from investigations with counters[3] and the values of σ_e measured in Ref. 1.

The calculated curves are in satisfactory agreement with experimental results and are close to the curves calculated by the authors of that reference using an optical nucleon model.

The differential cross sections obtained by Cork et al.[4] for E_p = 2.24, 4.40, and 6.15 Bev make it possible to obtain several values of σ_t^2 and R^2 for each energy, using formula (2). Curves of $d\sigma(\vartheta)/d\omega$ calculated using average valu $\overline{\sigma_t^2}$ and $\overline{R^2}$ are in satisfactory agreement with the experimental data; the value σ_t = 44 millibarns for E_p = 2.24 Bev is also in satisfactory agreement with known results on the total cross section of the p — interaction. Linear extrapolation of the cross section σ_i of the inelastic processes and of the paramete R into the energy region up to 10 Bev was used to calculate the angular distribution of elastic p — p sca tering for 8 and 10 Bev (see table.).

E_p lab. sys., Bev	$R\times10^{-14}$cm	σ_t, mb	σ_e, mb	σ_i, mb	$d\sigma(\vartheta)/d\omega$, mb/ster	$1-\beta_l$
1.5	6.9±0.6	47.2±0.9*	20±2**	27**	25.6 exp $(-4.3\ \vartheta^2)$	1.7 exp $(-0.11\ l^2)$
2,24	6.6±0,8	44.1±3.7	17,9±5.1	26,2±6.4	33,5 exp $(-5.9\ \vartheta^2)$	1.6 exp $(-0.09\ l^2)$
2,75	6.7±0.8	41±1*	15±2**	26**	35.5 exp $(-7.5\ \vartheta^2)$	1.5 exp $(-0.07\ l^2)$
4,40	6.9±0.3	33.9±1.8	9.7±1,5	24.2±2.7	38,8 exp $(-13\ \vartheta^2)$	1,1 exp $(-0,04\ l^2)$
6.15	7.2±0,3	31,3±1.5	7,5±1.5	23.8±1.8	46.1 exp $(-19\ \vartheta^2)$	1.0 exp $(-0.03\ l^2)$
8	7.45	30.0	6.4	23.5	54.6 exp $(-27\ \vartheta^2)$	0.9 exp $(-0.02\ l^2)$
10	7.75	28.5	5.4	23.1	62.2 exp $(-36\ \vartheta^2)$	0.8 exp $(-0.01\ l^2)$

*Taken from Ref. 3 **Taken from Ref. 1

It is seen from the table that R increases almost linearly with the energy, starting with 2.24 Bev, bu can be assumed constant within experimental accuracy. If R is really constant, it is possible to determine $d\sigma(\vartheta)/d\omega$ by measuring σ_t and vice versa.

The results of the analysis show that the assumptions made by Belen'kii are also true for p — p scattering at $E_p \geq 1.5$ Bev, since the resultant formulas give satisfactory agreement with the experimental data. The values of σ_t, σ_e, and σ_i, calculated for 2.24, 4.40, and 6.15 Bev on the basis of this analysis, can obviously be used as preliminary data.

The authors are grateful to I. V. Chuvilo for valuable discussions and attention to the work.

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PARAMAGNETIC RESONANCE OF NEW ORGANIC RADICALS

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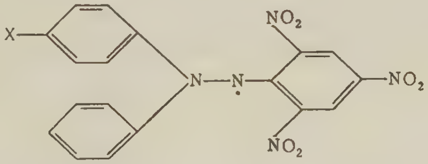
THE narrow line-width of the resonant absorption curve of the organic radical α,α-diphenyl β-picryl hydrazyl has been explained by exchange interaction of the odd electron in the $-N-N$ group.[1] Direct confirmation of the existence of the exchange interaction was obtained from studies of hyperfine structure of this absorption line.[2]

We undertook a set of experiments with an object to investigate the influence of substitutes X in para-position of the α-phenyl ring on exchange interactions in organic radicals. Para-fluoro-, para-chloro-, para-bromo-, and para-methoxy-derivatives of hydrazyl were investigated. Three of these (the para-fluoro-, para-chloro-, and para-methoxy-derivatives) were obtained first.



The line shape of the paramagnetic resonant absorption,[3] characterized by the ratio of fourth to the second moment of the resonant curve, M_4/M_2, served to determine the magnitude of exchange interactions. This ratio was obtained by numerical integration of the experimental absorption curve. The moments were computed separately for the left and right half of the curve and average values were taken.

Substitute X	g-factor	M_4/M_2	ΔH	A
H	2,0042 ± 0.0004	1.43±0.02	1.0 ± 0.15	300
Cl	2.001 ± 0,001	1,42±0.02	1.2 ± 0.15	290
Br	2,002 ± 0,002	1.40±0.02	2.2 ± 0.15	170
OCH_3	2.000 ± 0.002	1.30±0.02	2.6 ± 0.2	120
F	2.000 ± 0.004	small	4.1 ± 0.5	20

Measurement of the paramagnetic resonance absorption was performed using Zavoiskii's grid-current method.[4] To obtain absorption curves without admixture of dispersion, the generator circuit was coupled very weakly to the measuring coil. The generator frequency was varied in a fairly narrow region about a resonant frequency of 621 Mcs and was measured by a type 3003 heterodyne detector with an accuracy of 0.005%. A signal modulated at 224 cps was fed through a broad-band amplifier with a bandwidth 30 cps to 10 Mcs into an oscilloscope or into a narrow-band amplifier of bandwidth 5 cps and a synchronous detector with meter. Absorption curves were photographed from the screen of the oscilloscope. The narrow-band system was used for observing the width of the absorption curve between the points of steepest slope. The measured quantities were the maximum values A of the absorption curves, which yield values proportional to χ''. All the samples investigated contained equal molar amounts of radicals. The field was calibrated against the known resonance of α,α-diphenyl-picryl-hydrazyl, whose g-factor was taken to be 2.0042 ± .0004.[1]

The results are given in the table. It is seen that all the quantities, except the g-factor, which differs only slightly from 2, change gradually from H to F. The exchange interaction thus diminishes in the transition from H to F.

Visible and ultraviolet spectra were also measured for all compounds, using SF-2 and SF-4 equipmen In the visible part of the spectrum all the radicals investigated show absorption maxima at 520 mμ, and the maximum absorption decreases in the same manner as at radio frequencies. No sharp maxima were found in the ultraviolet spectral region.

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ASYMPTOTIC SELECTION RULES FOR BETA DECAY OF DEFORMED NUCLEI

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It was shown in a number of investigations[1-6] that for particle transitions in deformed nuclei one mus take into account not only the selection rules for the total angular momentum I and parity π but also th selection rules for asymptotic quantum numbers Λ, Σ, and also N and n_z (or n_z and $n_\perp$) introduce by Nilsson.[7] Λ and Σ are quantum numbers of the projection of orbital and spin angular momenta on th elongation axis of the nucleus (the z-axis); Λ + Σ = Ω, where Ω is the quantum number of the projection of the total angular momentum o the particle on the z-axis; N is the principal quantum number of the oscillator, and n_z an $n_\perp$ are the quantum numbers of nuclear oscillations along the z-axis and in the plane norm to the z-axis respectively.

Type	Matrix Element	$k=\Delta\Omega=\Delta I$	$\Delta\Lambda$	$\Delta\Sigma$	Δn_z	ΔN
S, V	$\int y_{\lambda k}(\mathbf{r})$	$\pm\lambda$	$\pm\lambda$	0	0	$\lambda, \lambda-2, \ldots -\lambda$
V	$\int y_{\lambda k}(\nabla)$	$\pm\lambda$	$\pm\lambda$	0	0	$\lambda, \lambda-2, \ldots -\lambda$
T, A	$\int y_{\lambda+1k}(\sigma)$	$\pm\lambda$ $\pm\lambda$ $\pm(\lambda+1)$	$\pm(\lambda-1)$ $\pm\lambda$ $\pm\lambda$	±1 0 ±1	±1 0 0	$\lambda, \lambda-2, \ldots -\lambda$
T, A	$\int y_{\lambda k}[\sigma\mathbf{r}]$	$\pm\lambda$	$\pm(\lambda-1)$ $\pm\lambda$	±1 0	±1 0	$\lambda, \lambda-2, \ldots -\lambda$
T	$\int y_{\lambda k}[\sigma\nabla]$	$\pm\lambda$	$\pm(\lambda-1)$ $\pm\lambda$	±1 0	±1 0	$\lambda, \lambda-2, \ldots -\lambda$

In this work selection rules for asymptotic quantum number N, n_z, Λ, Σ are establishe for β-transitions of an arbitrary order of forbiddenness (λ > 1) for different types of interactions.. (Allowed and singly-forbidden transitions were considered by Alaga.[1]) The table gives the results of the calculations. The firs column in the table shows the type of interaction — S, V, T, or A. For all transitions except λ = 1 the pseudoscalar interaction gives only small corrections to the matrix elements of transitions of (λ − 2)-fold forbiddenness. The second column gives the matrix elements of interaction operators in a non-rel tivistic approximation. Spherical representation is chosen.[8] Here

$$y_{\lambda k}(\mathrm{x}) = (i/\lambda\hbar)\,\mathrm{x}\{\mathrm{p} y_{\lambda k}(\mathrm{r})\}, \qquad \mathrm{x} = \sigma,\ \nabla,\ [\sigma\mathrm{r}],\ [\sigma\nabla], \qquad y_{\lambda k}(\mathrm{r}) = r^\lambda Y_{\lambda k}(\vartheta, \varphi).$$

The third column gives the selection rules for the total angular momentum I. Since in a strongly deformed nucleus the integral of motion is the projection of the total angular momentum on the elongation axis, we let ΔI = ΔΩ = k. Thus the rotational and the K-forbidden modes are excluded. Next two columns give the selection rules for Λ and Σ. These, of course, are related to the data of the preceding column (the condition ΔΛ + ΔΣ = ΔΩ must be satisfied). Let us note that for scalar or vector interactions only those transitions can take place for which the projection of the spin of the particle on the elon gation axis of the nucleus does not change. In this respect such β-transitions are analogous to electric multipole transitions. For tensor and pseudo-vector interactions transitions can occur with and without the change of the spin projection of the particle on the z-axis; this is analogous to magnetic multipole transitions. The fourth column gives the selection rules for n_z. In order to obtain these (as in the case of Λ, Σ) the wave functions of the nucleons of a strongly deformed nucleus must be used. The last column gives the selection rules for N. Selection rules for N and n_z are related to each other, as in the case for electromagnetic transitions.[6]

We emphasize that the selection rules for n_z, Λ, Σ are approximate. They hinder β-transitions rather than forbid them.

I should like to express by gratitude to L. A. Sliv and S. V. Izmailov for their interest in this work and for valuable advice.

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POLARIZATION OF 635 MEV PROTONS BY DEUTERONS IN QUASI-ELASTIC p — p SCATTERING

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EXPERIMENTS on the study of the interaction of high energy nucleons with nuclei show that in most cases the incident nucleon interacts with the individual nucleons of the nucleus. This is particularly evidenced by experiments on the quasi-elastic p — p and p — n scattering by nuclei,[1-4] carried out with unpolarized proton beams. The differences observed in these investigations between elastic and quasi-elastic scattering are explained by the fact that the latter is produced by nucleons moving inside the target nucleus. If polarized proton beams are used it is possible to make still another comparison between elastic and quasi-elastic scattering. Bradner and Donaldson[5] showed that at 285 Mev the polarization in quasi-elastic p — p scattering by Li, Be, and C nuclei has an angular dependence analogous to the angular dependence of polarization in elastic p — p scattering, but much smaller in magnitude than the latter. Thus, the polarization in quasi-elastic p — p scattering by Be amounts to only approximately 40% of the polarization in elastic p — p scattering. Such a reduction in polarization in quasi-elastic p — p scattering was attributed[6] to distortion of polarization in purely-elastic p — p scattering, caused by intra-nuclear motion of the target nucleons. Since the role of this motion becomes less important at greater energies, one can expect the polarization in quasi-elastic p — p scattering to approach the value of the polarization in elastic p — p scattering with increasing energy. This was actually observed by Mescheriakov, Nurushev, and Stoletov[7,8] in the scattering of 635-Mev protons by Be. Here the quasi-elastic polarization was 85% of the elastic one. One can expect that a scattering by nuclei with smaller binding energies, for example by deuterons, this difference will be even less. Chamberlain et al.[9] found that the polarization and quasi-elastic p — p scattering by deuteron at 312 Mev in the range 60 — 90° (c.m.s.) does not differ from the polarization in elastic p — p scatterin

The author measured the polarization of protons in quasi-elastic p — p scattering by deuterons at energies 635 ± 15 Mev. A proton beam from the synchrocyclotron o the Joint Institute of Nuclear Research with polarization 58 ± 3%, described in Ref. 7, was used. Th measurements were made by the counter-telescope method, and the angular resolution of the apparatu was 6°. Heavy water was used as the target. The background due to oxygen amounted to 30% of the counting rate and was determined by scattering polarized protons by the H_2O equivalent of the number of target nuclei under conditions excluding the counting of elastic p — p scatterings. The results obtained are given in the third line of the table. Only statistical errors are indicated here. For compariso the fourth line of the table gives data on polarization in elastic p — p scattering, obtained in Ref. 8, whil the fifth line gives data on quasi-elastic p — p scattering by beryllium, obtained in Ref. 7. It is seen from the table that at 635 Mev there is no difference in the polarization in elastic and quasi-elastic p — p scattering by deuterons, within the limits of experimental error. Thus, for a given energy, the intra-nuclea motion of the proton in the deuteron does not lead to a difference between the polarizations in quasi-elastic p — p scattering by deuterons and the polarization in free p — p scattering. Bearing this result in mind, one can expect that in this energy range the polarization in quasi-elastic p — n scattering by deuterons will equal the polarization in free p — n scattering. This is of interest, since the observation of the quasi-elastic p — n scattering by deuterons is one of the methods of studying the polarization effects in n — p interactions.

Values of Polarization at 635 Mev, Percent;
P_D — quasi-elastic p — p scattering by deuterons
P — elastic p — p scattering
P_{Be} — quasi-elastic p — p scattering by beryllium

θ lab. sys.	18°	24°	30°	36°	41°	52° (30°)
θ c.m.s.	41,2°	54,5°	67,5°	80,1°	90°	
P_D	43.5 ± 3.0	39.0 ± 3.8	30.2 ± 4.0	17.2 ± 4.5	—0.8 ± 4.5	—29.5 ± 3.5
P	42.4 ± 2.9	35.7 ± 2.3	27.9 ± 2.7	16.7 ± 2.6	—1.6 ± 2.5	
P_{Be}	35.4 ± 4.7		21.4 ± 3.1	11.2 ± 3.6	—1.0 ± 3.5	

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CONCERNING A SYMMETRY PROPERTY OF THE NEW GELL-MANN THEORY

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RECENTLY Gell-Mann[1] has proposed a new theory of the interaction between elementary particles. In this theory all baryons have spin ½, and the same mechanical mass and parity. They form a supermultiplet which is split up only when moderately-strong interactions occur by way of K mesons. The Hamiltonian of the interaction with π mesons is written in the following way:

$$H_\pi = ig\{(\bar{p}\gamma_5 p - \bar{n}\gamma_5 n)\pi^0 + \sqrt{2}(\bar{p}\gamma_5 n\pi^+ + \bar{n}\gamma_5 p\pi^-) + (\bar{\Xi}^0\gamma_5\Xi^0 - \bar{\Xi}^-\gamma_5\Xi^-)\pi^0 + \sqrt{2}(\bar{\Xi}^0\gamma_5\Xi^-\pi^+ + \bar{\Xi}^-\gamma_5\Xi^0\pi^-)$$
$$+ (\bar{\Sigma}^+\gamma_5\Sigma^+ - \bar{Y}^0\gamma_5 Y^0)\pi^0 + \sqrt{2}(\bar{\Sigma}^+\gamma_5 Y^0\pi^+ + \bar{Y}^0\gamma_5\Sigma^+\pi^-) + (\bar{Z}^0\gamma_5 Z^0 - \bar{\Sigma}^-\gamma_5\Sigma^-)\pi^0 + \sqrt{2}(\bar{Z}^0\gamma_5\Sigma^-\pi^+ + \bar{\Sigma}^-\gamma_5 Z^0\pi^-)\}. \quad (1)$$

(see Ref. 1 for the notation). In our note we shall show that in addition to the so-called global symmetry discussed by Gell-Mann, there is another symmetry property of his theory even in the absence of K coupling.

In the Hamiltonian (1) we make the substitutions

$$p \to \Xi_c^-,\ n \to \Xi_c^0,\ \Xi^- \to p_c,\ \Xi^0 \to n_c,\ \Sigma^+ \to \Sigma_c^-,\ Y^0 \to Z_c^0,\ \Sigma^- \to \Sigma_c^+,\ Z^0 \to Y_c^0,\ \pi^+ \to \pi^+,\ \pi^- \to \pi^-,\ \pi^0 \to -\pi^0, \quad (2)$$

where A_c denotes the charge-conjugation operator of the field of particle A. After this substitution the Hamiltonian takes the form

$$H_\pi = ig\{(\bar{\Xi}_c^0\gamma_5\Xi_c^0 - \bar{\Xi}_c^-\gamma_5\Xi_c^-)\pi^0 + \sqrt{2}(\bar{\Xi}_c^-\gamma_5\Xi_c^0\pi^+ + \bar{\Xi}_c^0\gamma_5\Xi_c^-\pi^-) + (\bar{p}_c\gamma_5 p_c - \bar{n}_c\gamma_5 n_c)\pi^0 + \sqrt{2}(\bar{n}_c\gamma_5 p_c\pi^+ + \bar{p}_c\gamma_5 n_c\pi^-)$$
$$+ (\bar{Z}_c^0\gamma_5 Z_c^0 - \bar{\Sigma}_c^-\gamma_5\Sigma_c^-)\pi^0 + \sqrt{2}(\bar{\Sigma}_c^-\gamma_5 Z_c^0\pi^+ + \bar{Z}_c^0\gamma_5\Sigma_c^-\pi^-) + (\bar{\Sigma}_c^+\gamma_5\Sigma_c^+ - \bar{Y}_c^0\gamma_5 Y_c^0)\pi^0 + \sqrt{2}(\bar{Y}_c^0\gamma_5\Sigma_c^+\pi^+ + \bar{\Sigma}_c^+\gamma_5 Y_c^0\pi^-)\}. \quad (3)$$

The following relations exist between the field operators A and B:[2]

$$\bar{A}B = \bar{B}_c A_c,\ \bar{A}\gamma_\mu B = -\bar{B}_c\gamma_\mu A_c,\quad \bar{A}\gamma_\mu\gamma_\nu B = -\bar{B}_c\gamma_\mu\gamma_\nu A_c,\quad \bar{A}\gamma_5\gamma_\mu B = \bar{B}_c\gamma_5\gamma_\mu A_c,\ \bar{A}\gamma_5 B = \bar{B}_c\gamma_5 A_c. \quad (4)$$

With the help of (4) we can satisfy ourselves that the Hamiltonian (3) agrees with (1). In the same way it can be shown that the Hamiltonian of the interaction of baryons with photons

$$H_\gamma = -ie\{\bar{p}\gamma_\mu p - \bar{\Xi}^-\gamma_\mu\Xi^- + \bar{\Sigma}^+\gamma_\mu\Sigma^+ - \bar{\Sigma}^-\gamma_\mu\Sigma^-\}A_\mu \quad (5)$$

is invariant under the substitution (2), with the addition of the substitution $A_\mu \to A_\mu$. This symmetry property of the Gell-Mann theory provides us with selection rules for certain processes. Let us consider processes in which there are only π mesons and photons in the initial and final states. For every Feynman diagram there will be another diagram in which the lines of the inner baryons are replaced by the propagation lines of the baryons appearing in the right-hand sides of the substitution (2). For example, proton lines are replaced by the lines of the baryon described by the field operator Ξ_c^- (i.e., by the lines of an anti Ξ^--particle), antiproton lines by the lines of a Ξ^--particle, etc. If the number of outer π^0 mesons is odd, then the matrix elements of these two graphs will cancel each other when they are added. Consequently in the new Gell-Mann theory the process of the decay of a π^0 meson into two γ-rays takes place only through a K interaction.

If it is assumed that β interactions of baryons also have Gell-Mann's global symmetry, then we can obtain a few other selection rules. Consider, for example, a tensor β interaction:

$$H_\beta = g_\beta\{\bar{p}\gamma_\mu\gamma_\nu n + \bar{\Xi}^0\gamma_\mu\gamma_\nu\Xi^- + \bar{\Sigma}^+\gamma_\mu\gamma_\nu Y^0 + \bar{Z}^0\gamma_\mu\gamma_\nu\Sigma^-\}\bar{e}\gamma_\mu\gamma_\nu\nu.$$

It is clear that this Hamiltonian changes sign under the substitution (2). Thus the matrix elements for the β-decay of a π^+ meson, that is

$$\pi^+ \to e^+ + \nu + n\gamma,\ n = 0,\ 1,\ \ldots$$

through a virtual baryon-pair, cancel each other in pairs. These processes also are allowed only through a K interaction.

If only the interactions of π mesons with nucleons are taken into account, then the theoretical probabilities of the processes

$$\pi^0 \rightarrow 2\gamma \text{ (Ref. 3) and } \pi^+ \rightarrow e^+ + \nu + \gamma \text{ (Ref. 4)},$$

obtained from perturbation theory, are greater than those observed. The considerations above indicate that it is possible to reduce the discrepancy between theory and experiment by allowing also for the interactions of π mesons with all baryons.* Of course, without a study of the interaction with a K meson we are still unable to say with certainty that the new Gell-Mann theory gives better agreement with the experimental data for these processes.

In conclusion I wish to thank Professor M. A. Markov, Professor Khu Nin, V. I. Ogievetskii, and M. I. Shirokov for their interest and for discussions of the work presented here.

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*This same result for the decay process $\pi^0 \rightarrow 2\gamma$ was also obtained by Gell-Mann.[1] An analogous result was obtained earlier by Kinoshita.[3]

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Translated by W. M. Whitney
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SUPERCONDUCTIVITY OF BERYLLIUM FILMS CONDENSED ON A COLD BACKING

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It is known that bismuth films obtained by condensation on a backing cooled with liquid helium display superconductivity.[1] Until now this was the only case of observed superconductivity induced in such a way in a metal which displays no superconductivity in the ordinary bulk state to temperatures on the order of 10^{-2} degrees K.

This letter reports on the superconductivity of beryllium films obtained in the above method.

The procedure for obtaining the films was analogous to that used in many investigations for the study of superconductivity of metallic films.[1,2] The apparatus consists essentially of an evacuated glass bulb, containing the evaporator in the form of a small tungsten helix carrying a piece of the evaporated metal. When the metal is evaporated the bulb is immersed in liquid helium. The low power of the evaporator (several watts) permits condensation of the layer with an insignificant temperature rise in the film above the bath temperature. Electrodes for the measurement of the electric conductivity of the resultant films are fused into the walls of the bulb.

The measurements were made with several films. The thickness of the measured films was approximately 10^{-6} cm. It is interesting to note that the only stable films were those thinner than 10^{-5} cm, and further increase in thickness caused them to crack.

Fresh films already display superconductivity at 4.2° K. An exact determination of the transition temperature has not yet been made, but a rough extrapolation of the dependence of the critical current on the temperature yields a value of approximately 8° K. As in the case of bismuth, the properties of the film vary greatly upon heating. Heating even to a temperature of liquid hydrogen transfers the superconductivity of the beryllium film into the liquid-helium temperature range.

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COMPARISON OF CALORIMETRIC AND IONIZATION MEASUREMENTS OF THE ENERGY FLUX OF SYNCHROTRON GAMMA RAYS

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THE comparison of the different methods of measurement with the identical spectrum is of interest not only for the determination of the energy flux of the γ-rays but also is useful in relating data obtained from ionization measurements to the actual energy absorption in matter. According to the literature,[1] various methods give results that differ up to 25%.

The energy flux in the γ-ray beam of the 85-Mev synchrotron of the Leningrad Physico-Technical Institute was determined both with a calorimeter and with ionization chambers. The energy in the γ-ray beam required to produce a charge of one Coulomb in a special copper ionization chamber was determined by both methods. The copper chamber served as a standard.

In the calorimetric measurement the γ-rays were absorbed in a lead cylinder. The temperature change of the cylinder was determined with a thermistor which had a temperature coefficient of about −6% at 20° C. The cylinders had a diameter of 5.5 cm and lengths of 11 and 4 cm respectively. The correction for incomplete absorption of the γ-rays in the cylinders was obtained from the transition curves of lead. The final results (at 20° C and 760 mm Hg) are given in Table I. They were obtained with a beam diameter of 3 cm. The energy losses due to neutron emission[2] were also taken into account. The result obtained with the 4 cm and the 11 cm long cylinders differ by 2%. This difference lies within the limits of the errors. It indicates, however, that the actual absorption of energy in lead is not given completely by the depth dose curve.

TABLE I

Cylinder length cm	Percent fraction of absorbed energy according to transition curves	Energy for one coulomb of the standard, U_K, Mev/coulomb	Maximum error %
11	99.5	$4.65 \cdot 10^{18}$	4.0
4	82	$4.55 \cdot 10^{18}$	2.5

The energy flux of the γ-rays was also determined by the method of the depth dose curve.[3] In this method the ionization in a thin walled chamber is determined as a function of the thickness of absorbers placed in front of it. The depth dose curves for C, Al, Cu, and Pb were obtained. The curves reproduced in several runs. The exponential decrease at large depths agreed well with the minimum of the γ-ray absorption coefficient in the particular material. The energy was obtained from the area under the transition curves. The results, particularly for light elements, depend strongly on the choice of the ratio (averaged properly over the energy of the electrons)

$$\rho = (dE/dX)_Z/(dE/dX)_{\text{Air}}$$

where dE/dX is the energy loss of the electrons. The choice of ρ and the results of the determination by the method of transition curves relative to the calorimetric determination (U_Z/U_k) are shown in Table II. A possible error of ρ of 2.5% has been

TABLE II

Material	C	Al	Cu	Pb
ρ	0.840	0.820	0.740	0.610
(U_Z/U_K) %	96	90.3	91.6	84
Total error %	6.5	7.3	9.0	5.5

included in the total error shown. Evidently only the results for lead cannot be reconciled with the calorimetric results.

A similar comparison was performed with thick-walled chambers.[4] The ionization in an air-filled cavity inside a block made of different materials was measured. The sensitivity of this chamber was determined theoretically for each wall thickness and material used in the experiment. The accuracy of this calculation is ± 5% and the possible error in ρ of ± 2.5% enter into the total error of the method. The results of this determination are given in Table III relative to the calorimetric results.

TABLE III

Thickness of front wall	2 cm Al	12 cm Al	4 cm graphite	22 cm graphite
used ρ'	0.820	0.820	0.840	0.840
(U/U_K), %	105.0	104.0	102.0	92.4
Total error %	8	8.5	8	8.5

Thus the results of the calorimetric determination, the ionization measurement, and the depth dose curve agree (except for the case of lead) within the total errors. However, the errors of the method of the transition curves can be decreased to 3%. For this purpose at the present time the method of the determination of the transition curves is being improved and the accuracy of the knowledge of ρ is being increased. In order to increase the sensitivity of the calorimetric method a calorimeter is being worked on which will allow measurements at low temperatures.

The author thanks Z. Kovarzh and I. V. Lopatin for their help in performing the measurements and A. P. Komar for his interest in the work.

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ON THE QUESTION OF THE FLUCTUATIONS OF ELECTRON CONCENTRATION IN THE F-LAYER OF THE IONOSPHERE

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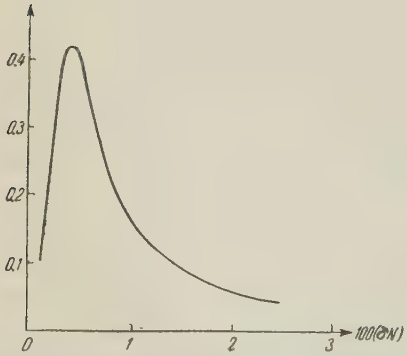
IT is known that the ionosphere is a statistically non-uniform medium, in which irregularities continually appear and disappear. The mechanism by which these irregularities arise is not yet known. At this time an investigation of the inhomogeneous structure of the ionosphere is necessary for an understanding of a number of processes that take place in the ionosphere.

Experimental investigations of the inhomogeneous structure of th ionosphere can be divided into two groups, the investigation of the fine structure of the ionosphere by vertical soundings, and the investigation by the forward scatter of uhf waves. Up to the present th forward scatter of uhf waves is in fact not explained. Therefore, great interest attaches to the mechanism of scattering of radio wave by inhomogeneities in the ionosphere, suggested by Al'pert.[1]



Distribution of values of δN

Using the results obtained by investigating the inhomogeneities in the structure of the ionosphere above Khar'kov by the method of vertical sounding during the winter of 1955 — 1956,[2] we have applied the Al'pert formula[1] to calculate the value of δN for ordinary waves.

The size of the inhomogeneities was taken to be 300 m in these calculations. The figure shows the distribution of the values of δN. The observed values of δN run from 0.1×10^{-2} to 2.5×10^{-2}, and the most frequent values are $\delta N = 0.3 - 0.5 \times 10^{-2}$. No noticeable altitude dependence of δN was found for the F-Layer.

In conclusion, the authors express their thanks to Professor Ia. L. Al'pert for giving them the opportunity to familiarize themselves with the theory worked out by him for the scattering of radio waves in the ionosphere prior to publication.

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PHOTOPRODUCTION OF STRANGE PARTICLES

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In this note, some questions connected with the photoproduction of strange particles are considered phenomenologically.

Recently, Peaslee[1] proposed a statistical model of the production of strange particles, in which he introduced the relative amplitudes α and β of the sojourn of the nucleon in the dissociated states (Λ, K) and (Σ, K), respectively. This model satisfactorily explains a number of experimental facts connected with the reaction $\pi + N \rightarrow Y + K$, if the value of $|\beta/\alpha|^2$ is taken as $1/3$. The photoproduction of strange particles on nucleons may serve as a verification of this model.

Let us consider, for example, the following reactions on hydrogen:

$$\begin{aligned} & \Lambda^0 + K^+, && \text{(1a)} \\ \gamma + p \rightarrow \; & \Sigma^+ + K^0, && \text{(1b)} \\ & \Sigma^0 + K^+. && \text{(1c)} \end{aligned}$$

It is not difficult to see that in the Peaslee model, reaction (1a) will have a cross section proportional to $|\alpha|^2$, and reaction (1c) will have a cross section proportional to $\frac{1}{3}|\beta|^2$. Thus, according to this model, K^+ particles "associated" with the Λ^0-hyperon would be expected to be more prevalent than K^+ particles produced together with the Σ^0 hyperon:

$$\sigma(c)/\sigma(a) = \tfrac{1}{3}|\beta/\alpha|^2 \approx 1/9. \tag{2}$$

Of course, the energy of the γ-quanta must be above threshold in order to avoid threshold effects.

Since the contribution of the Σ^+ hyperon currents to the cross section is small in comparison with the K^+ meson currents, reaction (1b) must also be considerably repressed with respect to reaction (1c). In the above consideration, we did not take account of the influence of the magnetic moment of the hyperon, which is valid for large energy of the γ-quanta. We note that according to the Peaslee model, a forward directionality of K^+ mesons (in the center of mass system) is expected.

As is well known, a simple classical consideration[2] leads to an excess of π^- mesons over π^+ mesons during their production respectively on neutrons and protons. A similar model is applied to the photoproduction of strange particles.

We will consider in particular the processes:

$$\gamma + p \rightarrow \Sigma^0 + K^+, \tag{3^0}$$

$$\gamma + n \rightarrow \Sigma^- + K^+. \tag{3$^-$}$$

While these processes are equivalent with regard to the interaction of the γ-quanta fields with the K^+ meson currents, they differ from one another with regard to the interaction of the γ-quanta with the hyperon currents.

Similarly to Ref. 2, the ratio of the cross sections of these processes is (in the laboratory system):

$$\frac{\sigma^0}{\sigma^-} = \frac{|(\mathbf{jA})_K|^2}{|(\mathbf{jA})_{\Sigma^-} + (\mathbf{jA})_K|^2}, \tag{4}$$

where $\mathbf{j}$ is the particle current, and $\mathbf{A}$ is the vector potential of the electromagnetic field. Applying conservation laws, we have (for nonpolarized γ-quanta):

$$\sigma^0/\sigma^- = [1 - (\varepsilon/m)(1 - v\cos\theta)]^2, \tag{5}$$

where ϵ is the total energy of the K^+ meson, v is its velocity, and θ is the angle between the directio of the γ-quantum and the K^+ meson. It is interesting that this equation coincides in form with the equation obtained by Brueckner and Goldberger[2] (m is the nucleon mass, $c = 1$).

If the magnetic interactions predominate over those of the currents, we obtain for σ^0/σ^- (compare Ref. 3):

$$\frac{\sigma^0}{\sigma^-} = \left(\frac{b}{a}\right)^2 \left[1 + \frac{(a\mu_p - b\mu_n)(\varepsilon/M)(1 - v\cos\theta)}{1 + b\mu_n(\varepsilon/M)(1 - v\cos\theta)}\right]^2, \tag{6}$$

where M is the mass of the Σ hyperon,

$$\frac{1}{a} = \mu_{\Sigma^0} - \frac{m}{M}\mu_p, \quad \frac{1}{b} = \mu_{\Sigma^-} - \frac{m}{M}\mu_n,$$

and μ is the magnetic moment of the respective particle in Bohr magnetons.

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217

ON THE USE OF TWO AUXILIARY FIELDS TO OBTAIN EMISSION STATES IN QUANTUM MECHANICAL AMPLIFIERS AND GENERATORS

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In order that a system of particles have negative absorption, it is necessary to produce an excess number of particles (active "molecules") in the upper energy level (m) of the given transition (m → q). A number of methods allows one to obtain active molecules by means of an auxiliary high frequency field with the auxiliary field having, in those cases when in the excitation process the distances between the levels remains unaltered, a frequency either greater than[1,2] of equal to[2,3] the transition frequency ω_{mq}. As is shown below, the last method — the method of "pulse inversion" — allows one by using two auxiliary fields to obtain active molecules with frequencies of the auxiliary fields being somewhat lower than the frequency of the main transition.*

We consider a system with four levels $m > p > n > q$ and with the equilibrium population of the i-th level given by N_i. We apply a pulse of the resonance field of frequency ω_{mn} which leads to a perturbation of the Hamiltonian of the form:

$$V = F^{+} \exp(-i\omega_{mn}t) + F^{-} \exp(i\omega_{mn}t).$$

Thanks to the periodic transition of the particles from the state m to the state n and back with the period[4] $2T_{mn} = \pi\hbar/|F_{mn}|$ if the duration of the pulse τ is equal to an odd number of half periods, the system will turn out to be inverted[3] (there will be N_n particles in the level m, and N_m particles in the level n). We assume that the relaxation time is large compared to τ. By applying a pulse of the second resonance field of frequency ω_{pq} which lasts an odd number of half periods T_{pq} we invert the population of the levels p and q. By choosing the field amplitudes in such a way that the same pulse length corresponds to the simultaneous inversion of the population of both pairs of levels, it is possible to act on the system with both fields simultaneously by a pulse of duration

$$\tau = (2k + 1)\, T_{mn} = (2l + 1)\, T_{pq}.$$

The two auxiliary fields enable us to obtain $N_n - N_p$ active molecules for the transition of frequency $\omega_{mq} > \omega_{mn}, \omega_{pq}$. Obviously $\omega_{mq} < \omega_{mn} + \omega_{pq}$. It is assumed above that any non-resonance influence of the field ω_{mn} on the transition pq and vice versa is excluded (either owing to the separation of the frequencies ω_{mn} and ω_{pq}, or owing to different polarization properties of these transitions). The frequencies ω_{mn} and ω_{pq} may coincide (for example, the two π-components right- and left-circularly polarized in the hyperfine structure spectrum; the transition $m \rightarrow q$ corresponds to the σ-component.† In this case it is sufficient to act on the system by a single auxiliary field which contains components of the required polarization (linearly polarized in the plane perpendicular to the magnetic field in the example discussed above) in order to obtain active molecules for the higher frequency transition.

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*We note, however, that this requires a special set of energy levels, transitions between which are allowed, and also that the frequency of at least one of the two auxiliary fields may be below ω_{mq} by not more than a factor two.

†The transition $(F, m \rightarrow F', m')$, σ-components $(3/2, 1/2 \rightarrow 1/2, -1/2)$ and $(3/2, -1/2 \rightarrow 1/2, -1/2)$, π-component $(3/2, 1/2 \rightarrow 1/2, 1/2)$, F and m are quantum numbers of the total angular momentum and of its projection (for a nuclear spin of 1).[5]

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LIMITING LAWS FOR SURFACE TENSION OF SOLUTIONS OF STRONG ELECTROLYTES

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I. Multivalent electrolytes of the asymmetrical type. In the surface layer of multivalent electrons of the asymmetrical type, unlike in those of the mono-monovalent electrolytes, the electric neutrality of each volume element is destroyed. This makes it necessary to take into account not only the potential due to the electrostatic representation of the ion with allowance for the screening action of the surrounding ions, but also the additional field due to the asymmetry of the electrostatic pattern. The potential of this field satisfies the equation

$$\Delta\varphi = -(4\pi/\varepsilon)\sum_i z_i e n_i, \tag{1}$$

where

$$n_i = n_{0i}\exp\{-U_i/kT\} \tag{2}$$

is the density of the ions of the i-th type in the surface layer. The potential energy U_i of the ions is

$$U_i = \frac{(z_i e)^2}{4\varepsilon kT}e^{-2\varkappa x} + z_i e\varphi \tag{3}$$

(z_i are positive and negative integers, e the electron charge, and $1/\kappa$ the Debye-Hückel radius[1]).

The distribution of the ions and of the charge density is determined by the following formulas

$$\frac{\delta n_i}{n} = \begin{cases} P_i\left(e^{-z_i^2 h/u} - 1\right) & \text{for } u \leqslant u_0 \\ -P_i z_i^2 h\,\dfrac{e^{-2u}}{u} - \dfrac{z_i P_i}{2}h\,\dfrac{\Sigma z_k^3 P_k}{\Sigma z_k^2 P_k}\left[e^{-u}\displaystyle\int_u^\infty \frac{e^{-v}}{v}\,dv - \right. & \\ \left. - e^{u}\displaystyle\int_{3u}^\infty \frac{e^{-v}}{v}\,dv - 2e^{-u}\left(\ln\frac{1}{\sqrt{3hz^{*2}}} + 1 - 2\gamma\right)\right] & \text{for } u \geqslant u_0 \end{cases} \tag{4}$$

$$\frac{\rho}{en} = \begin{cases} \displaystyle\sum_i z_i P_i\left(e^{-z_i^2 h/u} - 1\right) & \text{for } u \leqslant u_0 \\ -\displaystyle\sum_i z_i^3 P_i h\,\frac{e^{-2u}}{u} - \frac{h}{2}\sum_i z_i^3 P_i\left[e^{-u}\int_u^\infty \frac{e^{-v}}{v}\,dv - \right. & \\ \left. - e^{u}\displaystyle\int_{3u}^\infty \frac{e^{-v}}{v}\,dv - 2e^{-u}\left(\ln\frac{1}{\sqrt{3hz^{*2}}} + 1 - 2\gamma\right)\right], & \text{for } u \geqslant u_0, \end{cases} \tag{5}$$

where $\delta n_i = n_i(x) - n_{0i}$ are the deviations of the ion density in the surface layer from the density inside the electrolyte, and n is the density of the dissociated molecules. For convenience let us write $P_i = n_{0i}/n$ for the number of ions of the i-th type, formed upon dissociation of one molecule; u, for the distance in $1/\kappa$ units,

$$\ln z^* = \sum_i z_i^3 P_i \ln|z_i| \Big/ \sum_i z_i^3 P_i, \quad h = \frac{e^2}{4\varepsilon kT}\varkappa,$$

h is the small parameter of the theory (the ratio of the energy of the electrostatic interaction to the mean energy of thermal motion).

We note that the surface layer as a whole is electrically neutral, since the regions of positively and negatively charged cancel each other.

The following limiting law is derived for the change in the surface tension

$$\Delta\sigma = \frac{kT\varkappa^2}{16\pi}\left(\frac{3}{2} - 2\gamma + \ln\frac{2\varepsilon kT}{e^2\varkappa}\right) - \frac{kT\varkappa^2}{16\pi}\,\frac{\sum\limits_i (z_i e)^2 n_{0i}\ln(z_i e)^2}{\Sigma (z_i e)^2 n_{0i}} \tag{6}$$

($\gamma = 0.577$, the Euler constant). In practice it is more convenient to use this formula in the following form

$$\Delta\sigma = 39.7\left(\sum_i z_i P_i\right)\frac{m}{\varepsilon}\log[2.28\cdot 10^{-13}(\varepsilon T)^3/mz_0^4\,\Sigma z_i^2 P_i], \tag{7}$$

where m is the concnetration in moles/liter.

In the case of mono-monovalent electrolytes the resultant limiting law becomes the Onsager-Samaras limiting law.[2] The general formula they derived has no meaning, since the correction to the limiting law, contained in this formula, is of the same order as the quantities neglected by these authors.

2. Allowance for the dielectric constant of the external medium. In the case of mono-monovalent electrolytes the potential of the electrostatic field, produced by a fixed ion in its atmosphere, with allowance for the separation surface between the two dielectrics, is determined by the following equation

$$(\Delta - \varkappa^2)\varphi = -(4\pi/\varepsilon)e\delta(\mathbf{r}). \tag{8}$$

The solution of this equation is substantially more complicated than that for the case considered by Onsager and Samaras, owing to the fact that the dielectric constant of the external medium is not assumed to be zero. The adsorption potential (the potential energy of the interaction of the ion with its electrostatic image, with allowance for screening) turns out to be

$$W(z) = \frac{e^2}{2\varepsilon}\int\limits_{\varkappa}^{\infty}\frac{\lambda\varepsilon - \varepsilon'\sqrt{\lambda^2 - \varkappa^2}}{\lambda\varepsilon + \varepsilon'\sqrt{\lambda^2 - \varkappa^2}}\,e^{-2\lambda z}\,d\lambda. \tag{9}$$

In spite of the fact that it is necessary to calculate an integral of a non-elementary function (9), which is furthermore in exponential form, it is possible, using the smallness of the parameter of the theory, to obtain an expression for the change in surface tension Δs in terms of elementary functions

$$\Delta\sigma = \frac{e^2(\varepsilon - \varepsilon')n}{2\varepsilon(\varepsilon + \varepsilon')}\left[\ln\frac{2\sqrt{2}\cdot\varepsilon kT(\varepsilon + \varepsilon')}{\varkappa e^2(\varepsilon - \varepsilon')} + \frac{3}{2} - 2\gamma - \frac{1}{2}\left(\frac{\varepsilon + \varepsilon'}{\varepsilon - \varepsilon'}\right)^2\ln 2 + \frac{2\varepsilon^2}{(\varepsilon - \varepsilon')^2}\ln\frac{\varepsilon + \varepsilon'}{\varepsilon}\right]. \tag{10}$$

When $\epsilon' = 0$, Eq. (10) goes into the limiting Onsager-Samaras law.[2]

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Translated by J. G. Adashko
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REFINEMENT OF THE APPROXIMATION FOR SMALL ANGLES IN MULTIPLE-SCATTERING PROBLEMS

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To analyze the scattering of beams of charged particles by metallic foils it is necessary to solve the problem of the spatial and angular distribution of the density produced in the beam as a result of a large number of collisions. At a depth τ, the particle density in a monoenergetic (E > 10 Mev) beam $\psi(\mu, \tau)$

propagating at an angle θ relative to the axis τ ($\mu = \cos\theta$) is determined from the boundary problem

$$\mu \frac{\partial\psi}{\partial\tau} + \psi(\mu, \tau) - \frac{1}{2}\int\limits_{-1}^{+1} d\mu'\psi(\mu', \tau)\int\limits_{0}^{2\pi} \frac{d\varphi}{2\pi}\mathscr{F}(\cos\chi) = 0, \tag{1.1}$$

$$\cos\chi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi; \quad \psi(\mu, 0) = \delta(\mu - 1), \quad 0 \leqslant \mu \leqslant 1, \quad \psi(\mu, h) = 0, \quad -1 \leqslant \mu \leqslant 0. \tag{1.2}$$

The multiple-scattering distribution $\mathscr{F}(\cos\chi)$ is assumed known. Usually the problem is solved in the small-angle approximation, i.e., subject to the following simplifications: (1) the factor μ in front of the derivative (1.1) is assumed to be unity, (2) the conditions in (1.2) are replaced by the initial conditions $\psi(\mu, 0) = \delta(\mu - 1)$ at $-1 \le \mu \le 1$.

A numerical method proposed by Spencer[1] yields a solution in this approximation with an accuracy as high as desired for a wide class of problems, permitting representation of the function $\mathscr{F}(\cos\chi)$ by the sum[2]

$$\mathscr{F}(\cos\chi) = \sum_{i=1}^{N} A_i(1 + x_i - \cos\chi)^{-k_i}, \quad x_i \ll 1, \quad k_i = 1, 2, \ldots \tag{2}$$

It is noted in Refs. 3 and 4 that along with the good agreement between solutions in the small-angle approximation and experimental results at small angles, there is considerable discrepancy at large angles (more than 15% at $\theta > 6°$ for E = 15.7 Mev), which could not be explained by the authors within the framework of this approximation. Investigation of scattering at large angles is very important in connection with the study of the distribution of charge in the nucleus, of the character of the nuclear potential,[3] of phenomena connected with polarization, of differences in the scattering of positive and negative particles, etc.

Many computations were made to refine the small-angular approximation and to determine the region of its validity.[2]

1. Corrections for the small-angle approximation calculated by Spencer[1] were determined for the problem of scattering of 15.7 Mev electrons. In this problem

$$\mathscr{F}(\cos\chi) = x_0(1 + x_0 - \cos\chi)^{-2}, \quad x_0 = 1.56\cdot 10^{-6}, \quad h = 118.4.$$

This approximation is close to the Gauss function at small angles $\theta^2 \le x_0 h$, and tends to the function $4x_0h/\theta^4$ at larger angles.

We represent $\psi(\mu, \tau)$ as a sum $\psi_0 + \psi_1 + \psi_2$ where ψ_0 is the solution to the problem in the small-angle approximation, ψ_1 is the correction in the same approximation, i.e., the solution of the equation

$$\frac{\partial\psi_1}{\partial\tau} + \psi_1 - \frac{1}{2}\int\limits_{-1}^{+1} d\mu'\psi_1(\mu', \tau)\int\limits_{0}^{2\pi} \frac{d\varphi}{2\pi}\mathscr{F}(\cos\chi) = (1 - \mu)\frac{\partial\psi_0}{\partial\tau} \tag{3}$$

with initial condition $\psi_1(\mu, 0) = 0$ $(-1 \le \mu \le 1)$, while the correction ψ_2 is the solution of the inhomogeneous equation corresponding to (1.1) with the right half containing $(1 - \mu)\partial\psi_1/\partial\tau$. The boundary conditions for $\psi_2(\mu, \tau)$ are:

$$\psi_2(\mu, 0) = 0, \ 0 \leqslant \mu \leqslant 1; \ \psi_2(\mu, h) = -\psi_0(\mu, h) - \psi_1(\mu, h), \ -1 \leqslant \mu \leqslant 0.$$

The solution of problem (3) in the small-angle region is well represented by the Gauss function (in θ) multiplied by a certain polynomial in θ^2; the asymptotic expression in the large-angle region is $2x_0h/\theta^2$. The absence of a strong anisotropy in the inhomogeneities of the problem for $\psi_2(\mu, \tau)$ makes it possible to seek a solution in each of the intervals (0, 1) and (−1, 0) in the form of an expansion in Legendre polynomials $P_n(\mu)$ with $n \le 3$. The table gives the values of $\psi_1(\mu, h)$ for $\mu \ge 0$, obtained by numerical integration with Spencer's method, and the calculated results for $\psi_2(\mu, h)$ for $\mu \ge 0$.

$\cos\theta$	1	0.999	0.99	0.95	0.90
$\psi_1(\mu, h)$	0.1478	0.2349	0.02623	0.004384	0.001985
$2x_0h/\theta^2$		0.1940	0.01834	0.003668	0.001814
$\psi_2(\mu, h)$	0.2804	0.2788	0.2651	0.2071	0.1413

2. Several general results were estab-

lished for an arbitrary scattering law, the degree of anisotropy of which is determined by the smallness of the parameter

$$\varepsilon = \frac{1}{2}\int\limits_{-1}^{+1}(1-\mu)\,\mathscr{P}(\mu)\,d\mu,$$

i.e., by the smallness of the rms angle upon single scattering. The small-angle approximation is represented by the sum

$$\delta(\mu-1)\,e^{-\tau/\mu}+I_0(\mu,\tau),$$

where $I_0(\mu,\tau)$ is close to the Gauss function $\exp(-\theta^2/\overline{\theta^2})$, $\overline{\theta^2}\approx h$, at small angles $(\theta^2 \lesssim \epsilon h)$. If $\mathscr{P}(\cos\chi)$ is taken in the form (2), the asymptotic expression at large angles is

$$2^{\eta}A_{\eta}h\,\theta^{-2\eta},\qquad \eta=\min k_i.$$

The free term in Eq. (3) is a quantity of the order of ϵ. The function $\psi_1 = O(\epsilon h\psi_0)$ at small angles $(\theta^2 \lesssim h)$, and $\psi_1 = O(\theta^2\psi_0/2)$ at greater angles. Thus, at $\epsilon h \ll 1$, as expected, the correction ψ_1 is small in the region $\theta^2/2 \ll 1$. On the other hand, the comparatively large value of the correction $\psi_2(\mu,\tau)$ in this example limits the range of validity of the small-angle approximation to the inequality $\psi_0 \gg \psi_2$, i.e., $\theta^4 \ll \epsilon h$. Since in this problem $\epsilon = x_0 \ln x_0$, one would expect that the approximation here is good in the region $\theta \lesssim 1/10$ and is acceptable at $\epsilon h \ll 1$. To obtain solutions with a sufficient degree of accuracy at large angles it is proposed to use the interpolation method developed in Refs. 2 and 5.

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Translated by J. G. Adashko
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COMPARISON OF NEUTRON SPECTRA IN THE FISSION OF U^{233}, U^{235}, Pu^{239}

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THE spectra of fission neutrons from U^{235}, U^{233}, and Pu^{239} have been reported in a number of papers.[1-10] Measurements of the fission neutron spectrum from U^{235} (Refs. 1—8) are in satisfactory agreement with the semi-empirical formula of Watt.[3]

According to the data of Mukhin, Barkov, and Gerasimov,[8] the fission neutron spectra from U^{233} and Pu^{239} are the same as the spectrum from U^{235}, within experimental errors of 10—20%. The data of Nereson[9] and of Grandl and Neuer[10] indicate that the neutrons from Pu^{239} are somewhat harder than those from U^{235}.

This note presents a comparison of the neutron spectra from the fission of U^{233}, U^{235}, and Pu^{239}. Various neutron detectors were used.

The fission neutrons were obtained by irradiating samples of U^{233}, U^{235}, and Pu^{239} with thermal neutrons from a reactor. In the first series of measurements, the neutrons were detected by using the thresholds of the reactions $Pr^{141}(n, 2n)Pr^{140}$, $Al^{27}(n, p)Mg^{27}$, $P^{31}(n, p)Si^{31}$, and $Au^{197}(n, \gamma)Au^{198}$. To compare the intensities of the fission neutron sources, we used a fission camera with U^{233}. The irradiation took place inside a cavity 20 × 20 × 40 cm in the thermal column of the reactor.

In the second series of measurements, the neutrons were detected with fission cameras using U^{233}, U^{238}, and Np^{237}.

In the third series of measurements, the neutrons were detected with fission cameras using U^{238}, Np^{237}, and Th^{232}. A fission camera with U^{235} was used to compare the fast neutron fluxes.

The reactor power was controlled in all three series of measurements. The results obtained are shown in the table. These results include small (< 2%) corrections for inelastic neutron scattering in the converters. A_{233}, A_{235}, and A_{239} are the counting rates in the various detectors, normalized to unit flux of fission neutrons from the corresponding converter, indicated by the index subscript of A. Eeff is the effective threshold of the reaction.

Detector	A_{239}/A_{235}	A_{233}/A_{235}	E_{eff} (Mev)	Measurement Series
Pr^{141} (*n*, 2*n*) Pr^{140}	1.42 ± 0.08	1,30 ± 0.03	11.5	I
Al^{27} (*n*, *p*) Mg^{27}	1.13 ± 0.02	1,11 ± 0.03	4.6	I
P^{31} (*n*, *p*) Si^{31}	1.07 ± 0.03	1.07 ± 0.05	3.5	I
Th^{232} (*n*, *f*)	1.034 ± 0.017		1.6	III
U^{238} (*n*, *f*)	1.015 ± 0.013	1.02 ± 0.017	1.4	II
U^{238} (*n*, *f*)	1.04 ± 0.01		1.4	III
Np^{237} (*n*, *f*)	0,985 ± 0.01	1.01 ± 0.017	0.7	II
Np^{237} (*n*, *f*)	1.028 ± 0.014		0.7	III
Au^{197} (*n*, γ) Au^{198}	1.02 ± 0.03	1.01 ± 0.02	—	I

The data show that the fission neutrons from U^{233} and Pu^{239} are harder than those from U^{235}. The temperature differences between the fission fragments of U^{233} and Pu^{239} compared with those of U^{235} are estimated to be 0.04 ± 0.01 and 0.05 ± 0.01 Mev.

On the basis of models for the evaporation of neutrons from moving fragments, and assuming that the excitation energy of the fragment is connected with the mean number ν of neutrons evaporated through the relation $\Delta\nu/\Delta E = 0.120$ Mev^{-1} (Ref. 6), temperatures T of 1.02 ± 0.01 and 1.06 ± 0.01 Mev are found for the fragments of U^{233} and Pu^{239} respectively. The uncertainty in T corresponds to the uncertainty in ν.[11] The temperature $T = 1.00$ Mev was adopted for the fragments of U^{235}.

The difference between the average energy of neutrons from the spontaneous fission of Cf^{252} and those from U^{235} can be estimated using the values $\nu = 3.53 \pm 0.15$[12] and $\nu = 3.82 \pm 0.12$[13] for Californium. The result is a figure of 9 — 11%. This agrees well with the experimental value 8% found in Ref. 14.

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CONCERNING THE REALITY OF THE NONSTATIONARY MODEL OF THE INTERMEDIATE STATE

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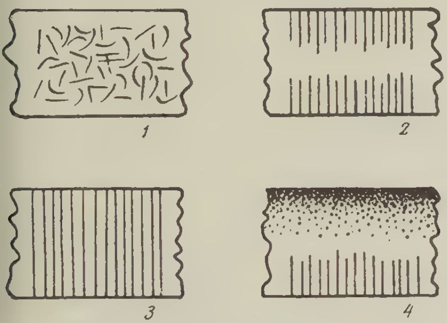
GORTER[1] has suggested a model of the intermediate state of superconductors with currents, in which continuous shifting of the boundaries between the superconducting and normal phases takes place. In the case of a cylindrical sample with current flowing parallel to the axis, this model leads to the formation of cylindrical regions and rings, moving and gradually constricting from the periphery to the axis of the cylinder. Under the joint action of a magnetic field perpendicular to the axis of the cylinder, and current, these regions, directed along the current and the field, move perpendicular to the axis of the cylinder.



Our observations of the intermediate state do not agree with the Gorter picture. Thus, B. Balashova, using a toroidal sample in which a superconducting current was excited by a field, observed an extremely sharp distribution of regions of normal and superconducting phases, having the form of layers situated perpendicularly to the axial line. However, this picture was obtained under nonstationary conditions, and to eliminate objections associated with this, we undertook experiments in which the picture of the distribution was observed with a constant current flowing through a cylindrical sample.

Two series of measurements were carried out: in the first series, the variable component of the magnetic field of the current on the surface of a monocrystalline tin sample (2 mm in diameter and 10 cm long) was measured using a bismuth microprobe. In the frequency range from 40 to 30,000 cps, no periodic or nonperiodic signals were observed with amplitudes exceeding 10 μv (noise level), which would have occurred with changes of the magnetic field (at $T = 3.5^\circ$ K and $H_K = 30$ oersted) of the order of 0.02 oersted. Also, no effect was observed under the joint action on the sample of the current and an external magnetic field.

In the second series of experiments, a sample 8 mm in diameter and 50 mm long was used. Magnetic powder was used to display the structure of the intermediate state. The following experiments were performed (see figure).

1. The sample was cooled to $T = 3.5^\circ$ K in a field exceeding H_c, and the picture of a disordered distribution of normal and superconducting regions was observed.

2. The sample was cooled to $T = 3.5^\circ$ K, after which a field exceeding $H_c/2$ was switched on. In this case the powder distribution was reminiscent of the pictures obtained by Balashova on a toroid, except that the layers perpendicular to the generator of the cylinder do not intersect in its central part. Similar pictures were observed by Schawlow[2] on plane samples.

3. The samples were cooled to $T = 3.5^\circ$ K in a magnetic field exceeding $H_c/2$. Then a current ~ 15% of critical was sent through the sample. After applying magnetic powder, a very sharp regular picture was observed of the distribution of regions of normal and superconducting phases, situated perpendicularly to the axis of the cylinder. In fields of 60, 75, and 90% of H_c, the distance between the regions varied from 0.4 to 1 mm, respectively.

4. With currents greater than 15% of I_c, and with relatively small transverse fields, the picture remained just as sharp, but regions of normal phase appeared in the parts of the cylinder at which the total field exceeded critical.

Thus, the structure predicted by Gorter did not occur experimentally.

The regularity of the distribution of the layers of normal and superconducting phases may undoubtedly be utilized for the measurement of the surface stresses of superconductors.

I thank Iu. V. Sharvin for discussions and D. I. Vasil'ev for assistance during the experiments.

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SPINS AND PARITIES OF THE HYPERFRAGMENT H_Λ^4 AND OF THE K MESON

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In this note we shall give a summary of the angular correlations in the cascades:

$$\pi^- + \mathrm{He}^4 \to H_\Lambda^4 + K,\ H_\Lambda^4 \to \mathrm{He}^4 + \pi,\ K \to \pi + \pi; \qquad (1)$$

$$K^- + \mathrm{He}^4 \to H_\Lambda^4 + \pi,\ H_\Lambda^4 \to \mathrm{He}^4 + \pi \qquad (2)$$

for several variants of spins in parities of H_Λ^4 and K.

We assume at first that the spin of the K meson is zero. If in cascade (1) the first reaction takes place near the threshold (kinetic energy of the π^- mesons in the laboratory is 620 to 640 Mev), then it is natural to assume that H_Λ^4 and K are formed predominantly in the s state if π (the product of the parities of π^-, He^4, H_Λ^4, and K) is $(-1)^i$ (i is the spin of the hyperfragment), and in the p state if $\pi = (-1)^5$ (in this case the production in the s state is forbidden). The distribution about the angle γ between the direction of the incident pions and the H_Λ^4 decay products (in the system where H_Λ^4 is at rest), given in the table, has been obtained under these assumptions. As can be seen, it is possible in principle to determine not only i, but also the products of the parities of H_Λ^4 and K (assuming that the product of the parities of the π^- and He^4 is -1).

Parity variant	Spin variant		
	0	1	2
$\pi = (-1)^i$	1	$3\cos^2\gamma$	$^5/_4\,(1-6\cos^2\gamma+9\cos^4\gamma)$
$\pi = -(-1)^i$	Forbidden reaction	$^3/_2\,(1-\cos^2\gamma)$	$^{15}/_2\,(\cos^2\gamma-\cos^4\gamma)$

Cascade (1) contains three reactions, and it therefore permits also the determination of the spin of the K meson by the Adair method[1] (true, if $k \neq 0$, the product of the parities of H_Λ^4 and K is no longer determined by the angular correlations). For this purpose one first selects such cases of the reaction $\pi^- + \mathrm{He}^4 \to H_\Lambda^4 + K$, in which the hyperfragment and the K meson make (in the center of mass system) small angles $\vartheta_f \approx 0$ with the direction of the incident pions (aligned with the z axis). If the H_Λ^4 decays associated with the $K \to \pi + \pi$ decays are further selected such that the pions make small angles $\vartheta \leq \Delta\vartheta$ with the same z axis, then the distributions about γ, which are chosen on the basis of this selection and which determine the spin i, will be the same as on the threshold in the variant $\pi = (-1)^i$ (see table). If one chooses instead the decays $K \to \pi + \pi$, associated with the H_Λ^4 decays along the z axis, it is possible to determine k. The correlation relative to the angle between the z axis and the direction of the decay products of K (in the system where the K meson is at rest) is given as a function of k in the same first line of the table.[1] The value of the permissible intervals of small angles ϑ_f and ϑ ($\Delta\vartheta_f$ and $\Delta\vartheta$) diminishes with increasing $\ell'_{\max}$ — the maximum important orbital momentum of the products of the reaction $K^- + \mathrm{He}^4$ $H_\Lambda^4 + K$ — and spin k respectively. If $\ell'_{\max} = 1$, we get $\Delta\vartheta_f \approx 20°$.

Cascade (2) was first proposed by Dalitz,[3] and Gell-Mann[4] gives a set of correlations for this cascade. If $k = 0$, these correlations have the same form as in the table, but relative to another angle θ, see Ref. 4. We emphasize here only that these formulas must be compared with the experimental distributions, obtained for the reactions $K^- + \mathrm{He}^4 \to H_\Lambda^4 + \pi^0$ without preliminary formation of a mesonic atom. If we verify somehow that the reaction took place "in flight," one can expect that at K-meson energies up to

20 Mev (but above 0.01 Mev) the He^4 nucleus captures the K meson in the s state [or in the p state, if $\pi = -(-1)^i$] by some specific (non-electromagnetic) short-lived forces.

At greater K-meson energies the correlation formulas are the same, but are valid only for reaction products making small angles with the direction of the incident K meson.

Cascade (2) does not permit determination of the spin of the K meson, and the effect of $k \neq 0$ on the discussed correlations relative to θ can be investigated only qualitatively. If $k \neq 0$, the correlation corresponding to a given i becomes less anisotropic, smoothes out, and thereby yields too low values of the hyperfragment spins.

I express my gratitude to Professor M. A. Markov for discussions.

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RADIATION FROM AN ELECTRON TRAVERSING A SLAB

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WHEN an electron moves in an inhomogeneous medium it radiates. If its velocity is greater than the phase velocity of light, it will also emit Cerenkov radiation. The radiation emitted by an electron moving at right angles to the boundary between two media was first investigated by Ginzburg and Frank[1] (see also Refs. 2 and 4). In the present paper we wish to communicate the results of calculations on the angular distribution of the radiation emitted by a charge moving perpendicularly to a slab placed in a vacuum.

The energy emitted into the half-space in front of the slab (z < 0) is as follows:

$$W = \frac{2e^2v^2}{\pi c^3}\int_0^\infty d\omega \int_0^{\pi/2} \sin^3\vartheta \cos^2\vartheta \, |A(\omega, \vartheta)|^2 \, d\vartheta, \tag{1}$$

$$\begin{gathered} A(\omega, \vartheta) = (\varepsilon' - 1)\,[e^{-ixd\omega/c}(1+\beta x)(x-y)(1-\beta^2-\beta x) + e^{-ixd\omega/c}(1-\beta x)(x+y)(1-\beta^2+\beta x) \\ - 2xe^{id\omega/v}(1-\beta\cos\vartheta)(1+\beta\cos\vartheta-\varepsilon'\beta^2)]\,[e^{-ixd\omega/c}(x-y)^2 - e^{ixd\omega/c}(x+y)^2]^{-1}(1-\beta^2\cos^2\vartheta)^{-1}(1-\beta^2x^2)^{-1}; \\ x = \sqrt{\varepsilon' - \sin^2\vartheta}. \quad y = \varepsilon'\cos\vartheta. \end{gathered} \tag{2}$$

The z axis is in the direction of the moving electron; d is the thickness of the slab, whose dielectric constant ϵ' may be complex (if there is absorption);* ϑ is the angle between the z axis and the direction of observation.

The angular distribution of the radiation behind the slab (in the half-space $z > d$) is obtained by replacing v by −v. If d is allowed to increase to infinity, and at the same time the absorption is taken to be finite, with $\mathrm{Re}\,(ix) > 0$, then the formula above reduces to the result of Ginzburg and Frank† for the radiation emitted by an electron flying into an infinite medium.

Analysis of the solution shows that for a nonrelativistic electron the angular distribution of the radiation is the same in the forward direction as it is in the backward direction. For a thin slab ($d \ll \omega\sqrt{\epsilon/c}$), the angular distribution of the radiation emitted by a nonrelativistic electron turns out to be the same as the angular distribution of the radiation from an oscillating dipole; in the most favorable case, i.e., for

$\omega d/v = \pi(2n + 1)$, the energy of the radiation is much greater than it is when only one boundary is present.

If the slab thickness is greater than the wavelength, interference maxima occur. These are due to multiple refractions of the radiation at the boundaries and interference between the radiation emitted at the first and second boundaries.

If the electron speed is such that $\epsilon\beta^2 > 1$, Cerenkov radiation will be emitted in the slab. It can be internally reflected, and then the characteristic maximum will not be observed. If the conditions for internal reflection are not satisfied, then there will be a sharp maximum at an angle ϑ given by

$$\beta^2(\varepsilon - \sin^2\vartheta) = 1, \tag{3}$$

The height of the maximum will depend on the Cerenkov radiation. The angle determined by (3) characterizes the Cerenkov cone, refracted at the boundary.

Quantitative comparisons between the theory of Cerenkov radiation and experiment have been made using slabs; some discrepancies were found. Our solution is exact, and a more reliable comparison with experiment can therefore be made. In addition, the process might be interesting as a microwave generator, but this awaits further investigations.

The author is grateful to V. L. Ginzburg, I. M. Frank, and B. M. Bolotovskii for discussions of his results.

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*Where the dielectric constant is written without a prime, the slab is taken to be transparent.

†The final results in Refs. 1 and 2 have a misprint in the sign.

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INFLUENCE OF ELECTROSTATIC FIELD ON THE ABSORPTION OF SOUND IN ROCHELLE SALT

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WE recently reported anomalous absorption of sound in Rochelle salt, observed by us at a temperature near its upper Curie point.[1] This absorption takes place for a transverse elastic wave propagating along the crystallographic z axis and polarized along the y axis. An examination of the piezoelectric properties of Rochelle salt shows that the shear strains connected with the propagation of this wave cause polarization of the crystal along its ferroelectric x axis. It is therefore important to investigate the absorption of the same waves in the case when a crystal is already polarized beforehand by an external electrostatic field directed along the same x axis.

The absorption of sound was measured by the pulse method. An oscillograph was used to record the amplitudes of the ultrasonic pulses passing through a Rochelle salt crystal. The pulse duration was 1.5 milliseconds and the oscillation frequency was 5 Mc.

Our experiments have shown that at temperatures above the Curie point, when the crystal loses its ferroelectric properties, absorption of sound is practically independent of the external field. At temperatures a few degrees below the Curie point, absorption of sound is weakly dependent on the external ield. Specifically, a field of intensity ≈ 600 v/cm changes the sound absorption by only several percent. This is apparently due to the fact that the external field intensities away from the Curie point become small compared with the molecular field of the spontaneous polarization inside the crystal.

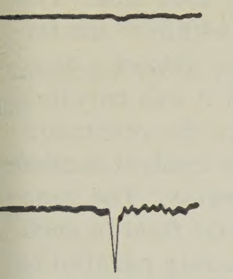

FIG. 1

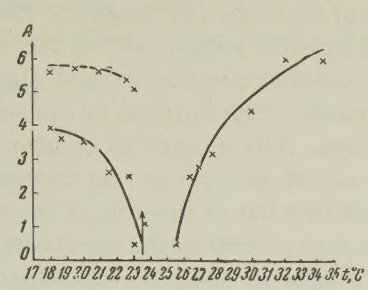

FIG. 2

However, at temperatures 0.1° to 0.2° below the Curie point, the influence of the electrostatic field on the absorption of sound in the Rochelle salt is very large. At these temperatures placing the crystal in an external field of the intensity indicated increases the amplitude of the sound wave passing through this crystal by a factor of several times ten. Figure 1 shows two oscillograms of sound pulses passing through a Rochelle salt crystal at a temperature $T - \Theta_c \approx 0.2°$. The upper oscillogram corresponds to the passage of sound through an unpolarized crystal. It shows only the zero sweep line of the oscillograph, washed out by the noise. The sound signal is not noticeable at this temperature, owing to the strong absorption of sound at this temperature. The lower photograph shows an oscillogram of a sound pulse under the same conditions, but after passing through a crystal polarized by an external field. The amplitude of the pulse is increased by approximately 50 times.

The polarization of the crystal by an external field also affects substantially the temperature dependence of sound absorption below the Curie point. In the absence of an external field, in the temperature range from 23° to 19° C, the absorption of sound diminishes monotonically with cooling of the crystal, while in a crystal polarized by an external field near the Curie point, the absorption of sound becomes practically independent of the temperature. With this, the amplitude of the sound signal passing through the polarized crystal depends on the temperature at which the external field was applied to the crystal. Figure 2 shows typical plots of the temperature dependence of the amplitude of the signal passing through the Rochelle salt. The solid and dotted curves pertain to the unpolarized and polarized crystal respectively. The high temperature branches of both curves are identical. The arrow on the diagram shows the jump in intensity of sound signal as a result of the superposition of an external electrostatic on the crystal.

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CYCLOTRON RESONANCE IN TIN AT 9300 Mcs

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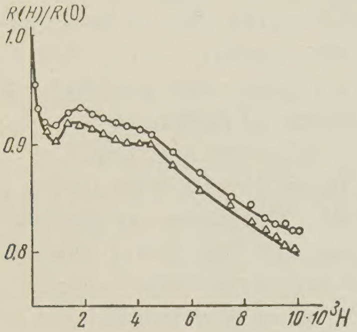
AZBEL' and Kaner[1,2] showed that cyclotron resonance should take place in a magnetic field parallel to the surface of a metal provided that $\delta \ll r \ll \ell$ (δ is the skin-depth, r the radius of curvature of the electron orbit in the magnetic field, and ℓ the mean free path of the electrons). In contradistinction to diamagnetic resonance in semiconductors,[3] cyclotron resonance in metals takes place not only when the frequency of the external high frequency field ω coincides with the Larmor frequency Ω but also at higher harmonics. In addition, one can only observe cyclotron resonance in fields that are strictly parallel to the metal surface.

To observe this phenomenon experimentally we undertook a measurement of the dependence of the active part of the surface resistance of tin on the strength of the constant magnetic field at 9300 Mcs. The sample we used was an electropolished single crystal of tin with a diameter of 0.8 mm; within 5° the tetragonal axis coincided with the axis of the sample. The specimen was arranged along the axis of a coaxial copper resonator operating in the fundamental TEM mode. The surface hf current in it was parallel to the axis of the specimen. The electrical connection of the resonator with the high frequency power generator was through the central conductor of the coaxial line leading out of the cap of the apparatus. The resonator with the specimen was placed in the electromagnetic field in such a way that the constant magnetic field H was approximately parallel to the specimen axis. The final adjustment of the position of the specimen parallel to the field was obtained by turning it in the horizontal and vertical planes at temperatures $T < T_c$ in a magnetic field lying between $\frac{1}{2} H_c$ and H_c, until the current in the detector output was maximum. The current through the detector decreases when the specimen is no longer parallel to the field because the superconducting specimen goes over into the intermediate state. An estimate of the change of the surface resistance was obtained from the magnitude of the deflection of the pointer of a dc galvanometer connected in the detector circuit. The measurement was carried out when the generator was working continuously.



In the figure we give the results of the measurements R(H)/R(0) [R(H) and R(0) are, respectively, the surface resistance of the specimen in the field and in zero field] at a temperature of 4.2° K (upper curve) and at 2° K (lower curve) in the case where the hf current is parallel to the constant field and to the tetragonal axis of the specimen and $\omega t_0 = 6$. This value of ωt_0 (ω is the angular frequency of the electromagnetic field and t_0 is the relaxation time of the electrons) refers to a temperature of 4.2° K and was determined from the residual resistance. It can be seen that there are two resonance minima on the curves, which increase in sharpness with decreasing temperature. For fields exceeding 4000 oersted the resistance decreases monotonically in agreement with theory. The position of the resonance minimum at H = 900 oersted and also the monotonic decrease of the resistance in strong fields is also in agreement with the experimental results of Fawcett[4] which were obtained with tin at 24,000 Mcs.

If we assume that the resonance minimum at H = 3600 oersted corresponds to the condition $\omega = eH/m^* c$ and hence gives us an estimate for the effective mass m^* of the electrons, it turns out that $m^* \approx m_0$, where m_0 is the mass of a free electron. It is interesting to note that an estimate of the effective mass of the electrons from the conductivity of tin obtained from experiments on galvanomagnetic phenomena[5] leads to approximately the same value.

The fact that there are only two resonance minima on the experimental curve, one of which is only a weak minimum, is apparently connected with the small value of ωt_0. This point of view is confirmed by the experiments at 2° K. Indeed, at this temperature the resonance sharpens appreciably with a blending of the minimum on the side of large fields, in accordance with theory. It is also possible that the resonance minima at $H_1 = 3600$ and at $H_2 = 900$ oersted are caused by electrons of different effective mass ($m_1 \approx m_0$ and $m_2 \approx 0.25\, m_0$). Finally, the resonance effect is possibly suppressed by the fact that the field is not strictly parallel to the surface of the specimen.

Turning the specimen over 90° relative to the constant field confirms roughly this picture of the phenomenon, as it produces a decrease in the depth of the minimum and a blending of the minimum at smaller fields. The cause of the decrease in sharpness of the resonance is apparently the decrease of the effective surface of the specimen parallel to the field and the blending of the minimum in the region of smaller fields is caused by the anisotropy of the effective mass of the electrons.

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